Congruences of lines in the projective space and systems of conservation laws

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16 August 2006 / Segovia

Outline

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Congruences of lines & their order
definitions

We work over the complex field $\mathbb{C}$.

Definitions

A congruence of lines B of $\mathbb{P}^n$ is a family of lines of dimension $n-1$.
Its order is the number of lines of B passing through a general point of $\mathbb{P}^n$.

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Congruences of lines
Notations

We put

$$\begin{array}{ccccc} \Lambda_b \subset \Lambda \subset B \times \mathbb{P}^n & \xrightarrow{f} & \mathbb{P}^n \supset \Lambda(b) \\ \downarrow \quad \downarrow p & & \\ b \in B & & \end{array} \tag{1}$$

where

$$\Lambda := \{(b, P) \mid P \in \Lambda(b)\},$$
$$\Lambda_b := p^{-1}(b),$$
$$\Lambda(b) = f(\Lambda_b).$$

$\Lambda(b)$ is the line corresponding to the point b, Λ is the incidence correspondence.

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Congruences of lines
Foci

Definitions

- The focal divisor of (Λ, B, p), $R \subset \Lambda$, is the ramification divisor of $f : \Lambda \to \mathbb{P}^n$.
- The schematic image of R via f is the focal locus $F = f(R) \subset \mathbb{P}^n$.
- The fundamental locus, Φ, is the set of points y such that

$$\dim f^{-1}(y) > \dim B + \dim p^{-1}(b) - \dim f(\Lambda),$$

i.e. Φ is the set of points y such that the fibre $f^{-1}(y)$ has dimension bigger than the general one.

$\Rightarrow \Phi \subset F.$

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General results on foci

Proposition ([Seg88],[CS92])

Either $\Lambda(b)$ is contained in F or $\Lambda(b)$ contains exactly $n-1$ foci which are foci for $\Lambda(b)$ (counting multiplicities).

The expected dimension of F is $n-1$. If $\dim(F)=n-1$, $\Lambda(b)$ is tangent to F at its (smooth) focal points.

Proposition ([DP04])

If $\dim(F)\leq n-2$, then the order of B is zero or one.

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Systems of conservation laws

Definitions

We work over $\mathbb{R}$. All functions are – at least – C^1.

Definitions

A system of conservation laws is a quasilinear system of first order PDE of the following type

$$\frac{\partial u_i}{\partial t} + \frac{\partial f_i(\mathbf{u})}{\partial x} = 0 \qquad i = 1, \ldots, n-1 \qquad (2)$$

(the f_i's are functions defined on a domain $\Omega \subset \mathbb{R}^{n-1}$, $\mathbf{u}(x, t)$ is a $(n-1)$-tuple of unknown functions).

A system is called hyperbolic (resp. strictly hyperbolic) if all the eigenvalues of the Jacobian matrix $Jf(\mathbf{u})$ are real (resp. real and distinct).

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Definitions

If the system (2) is strictly hyperbolic, the eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{n-1}$$

are called characteristic velocities.
The integral trajectories of every field of eigenvectors $\mathbf{v}_i$ are the rarefaction curves γ_i:

$$\dot{\gamma}_i(t) = \mathbf{v}_i(\gamma_i(t)).$$

Proposition

Given a system of conservation laws as before, through every $\mathbf{u} \in \Omega \subset \mathbb{R}^{n-1}$ there pass $n-1$ rarefaction curves.

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A strictly hyperbolic system (2) is called linearly degenerate if

$$L_i(\lambda_i) = 0 \quad \forall i$$

where L_i is the *Lie derivative* in the direction of $\mathbf{v}_i$.

A strictly hyperbolic system (2) is called a Temple system or a T-system if it is linearly degenerate and the rarefaction curves are straight lines in the coordinates $(u_1, \ldots, u_{n-1})$ (see [Tem83]).

T-systems naturally appear in the theory of associativity equations of $2D$ topological field theories, see [Dub96].

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Conservation laws & families of lines

Agafonov & Ferapontov

To a system of conservation laws (2) it is possible to associate ([AF96] and [AF99]) a $(n-1)$-parameters family of lines B in $\mathbb{P}^n$, defined by the parametric equations

$$\begin{aligned} y_0 &= \lambda, \\ y_i &= u_i\lambda - f_i(\mathbf{u})\mu, \qquad\qquad i = 1, \ldots, n-1 \\ y_n &= \mu, \end{aligned}$$

where

- $(\lambda : \mu) \in \mathbb{P}^1$ are coordinates on a line of B;
- $\mathbf{u} = (u_1, \ldots, u_{n-1})$ are affine coordinates on B;
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Congruences associated to T-systems

Pencils of lines

In general, the focal locus of a hyperbolic system of conservation laws is a hypersurface F and the lines of the family are tangent to F at its $n-1$ foci.

For a T-system the situation is different.

The family of lines B associated to a T-system is characterized by the property that through any point of its focal locus there is a pencil of lines of B; moreover $\dim(F) = n-2$.

A reciprocal transformation of the system corresponds to a projectivity of $\mathbb{P}^n$, and vice-versa. In particular the classification of the T-systems is equivalent to the study of those families of lines B.

By Proposition 2, B has order one.

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Open problems

T-systems and congruences

- The T-systems have been classified in [AF02] and [AF01] up to $n = 4$, it results that all the associated families of lines are linear congruences, i.e. linear sections of the Grassmannian, in particular they are algebraic. Is the same true in general?
- The families of lines associated to the T-systems are formed by pencils of lines. We would like to classify the (algebraic) congruences of lines having this property.
- The classification is not known for $n \geq 5$, not even for linear congruences.
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Linear congruences in $\mathbb{P}^5$

Construction

The focal locus of a general linear congruence in $\mathbb{P}^5$ is a Palatini threefold, a scroll over a smooth cubic surface S in $\mathbb{P}^3$ (see [O92], [FM02]).

S can be obtained as follows: let B be defined as $\mathbb{G}(1,5) \cap \Delta$, where Δ is a 10-dimensional linear space.

The dual variety of the Grassmannian is a cubic hypersurface - the Pfaffian - so S is naturally identified with the intersection of $\check{\mathbb{G}}(1,5)$ with the dual of Δ.

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Classifying the linear congruences in $\mathbb{P}^5$ is equivalent to describing all special positions of the 3-space $\check{\Delta}$ with respect to $\check{\mathbb{G}}(1,5)$ and to its singular locus, which is naturally isomorphic to $\mathbb{G}(3,5)$.

For instance, when S meets $\operatorname{Sing}\check{\mathbb{G}}(1,5) \cong \mathbb{G}(3,5)$ at a point, the focal locus acquires a linear irreducible component.

The cases when S splits are particularly interesting: the description of these congruences follows from a recent classification of the linear systems of 6×6 skew - symmetric matrices of constant rank 4, under the natural action of the projective linear group $\mathbb{P}GL_6$ ([MM05]).

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Linear systems of constant rank

Theorem

A line of 6×6 *skew -symmetric matrices of constant rank 4 is* $\mathbb{P}\mathrm{GL}_6$*-equivalent to one of the following*

$$\ell_g = \begin{pmatrix} 0 & 0 & a & 0 & b & 0 \\ & 0 & 0 & a & 0 & b \\ - & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \ell_s = \begin{pmatrix} 0 & 0 & a & 0 & b & 0 \\ & 0 & b & a & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

The space of such lines is irreducible of dimension 22, with an open orbit of general lines and a codimension one orbit of special lines.

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Theorem

- *The space of planes of 6×6 skew -symmetric matrices of constant rank 4 has 4 connected components, each of which is a $\mathbb{P}\mathrm{GL}_6$-orbit of dimension 26.*
- *There are no 3-spaces of 6×6 skew -symmetric matrices of constant rank 4.*

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The four 2-planes:

$$\pi_5 = \begin{pmatrix} 0 & 0 & 0 & a & b & 0 \\ & 0 & a & b & c & 0 \\ & & 0 & c & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \pi_g = \begin{pmatrix} 0 & 0 & 0 & 0 & a & b \\ & 0 & 0 & -a & 0 & c \\ & & 0 & -b & -c & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

$$\pi_t = \begin{pmatrix} 0 & 0 & a & b & c & 0 \\ & 0 & 0 & a & b & c \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \pi_p = \begin{pmatrix} 0 & 0 & 0 & a & b & c \\ & 0 & a & c & 0 & 0 \\ & & 0 & b & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

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Linear systems of constant rank

The four 2-planes:

$$\pi_5 = \begin{pmatrix} 0 & 0 & 0 & a & b & 0 \\ & 0 & a & b & c & 0 \\ & & 0 & c & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \pi_g = \begin{pmatrix} 0 & 0 & 0 & 0 & a & b \\ & 0 & 0 & -a & 0 & c \\ & & 0 & -b & -c & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

$$\pi_t = \begin{pmatrix} 0 & 0 & a & b & c & 0 \\ & 0 & 0 & a & b & c \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \pi_p = \begin{pmatrix} 0 & 0 & 0 & a & b & c \\ & 0 & a & c & 0 & 0 \\ & & 0 & b & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

Linear congruences in $\mathbb{P}^5$

Linear systems of constant rank

Under the Gauss map the 4 planes correspond to the 4 embeddings of the Veronese surface in $\mathbb{G}(1,5)$.

This gives also the classification of the linear spaces contained in the secant variety of the Grassmannian and disjoint from the Grassmannian.

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Linear congruences in $\mathbb{P}^5$

Classification

Surface S	Sing S	Focal locus F	Remarks
smooth		smooth Palatini scroll	F has moduli
irreducible with isolated singular points $P_1, \ldots, P_k$, $k \leq 4$	$P_i \notin \mathbb{G}(3,5)$	sing. Palatini scroll	F can be sing. at a point or along a line
	$P_i \in \mathbb{G}(3,5)$	the 3-sp. π_{P_i} is an irr. comp. of F	the residual meets π_{P_i} in a smooth quadric

Linear congruences in $\mathbb{P}^5$

Classification

Surface S	Sing S	Focal locus F	Remarks
irreducible ruled with double line r	$r \cap \mathbb{G}(3,5) = \emptyset$	sing. Palatini scroll	F singular along a quadric cone, $\check{\Delta}$ tangent to $\check{\mathbb{G}}(1,5)$ along r

Linear congruences in $\mathbb{P}^5$

Classification

S	$\operatorname{Sing} S$	F	Remarks
$\pi \cup Q$, π 2-plane, Q quadric	conic $\subset \pi \setminus \mathbb{G}(3,5)$	$L' \cup X$, $\deg X = 6$	L' parasitic
		$\Gamma' \cup Y$, $\deg Y = 5$	Γ' has a parasitic line as embedded comp.
		$Z_1 \cup Z_2$, $Z_1 = \mathbb{P}^1 \times \mathbb{P}^2$	lines of B are secant to Z_1 and Z_2
		$C(\mathcal{V}) \cup T$, $\deg T = 3$	lines of B are trisecant $C(\mathcal{V})$ meeting T

Linear congruences in $\mathbb{P}^5$

Classification

A component of the focal locus is parasitic if it is not met by a general line of the family B.

To apply this to the classification of Temple systems it will be necessary to refine it over the real field.

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