

# Smoothing of $K3$ carpets on Enriques surfaces

Miguel González<sup>1</sup>

joint work with

Francisco J. Gallego<sup>1</sup> and Bangere P. Purnaprajna<sup>2</sup>

<sup>1</sup>Departamento de Álgebra  
Universidad Complutense de Madrid

<sup>2</sup>Department of Mathematics  
University of Kansas

Satellite Conference on Algebraic Geometry, Segovia 2006



# Part I

## Introduction



## Setup

- $X \xrightarrow{\pi} Y$ , étale double cover,  $Y$  smooth Enriques surface,  $X$  smooth  $K3$  surface.

## Objects of interest

- $K3$  carpet  $\tilde{Y}$ : Cohen–Macaulay double structure on  $Y$ .
- $\tilde{Y}$  has the invariants of a smooth  $K3$  surface:

$$\omega_{\tilde{Y}} \simeq \mathcal{O}_{\tilde{Y}} \text{ and } H^1(\mathcal{O}_{\tilde{Y}}) = 0.$$



## Setup

- $X \xrightarrow{\pi} Y$ , étale double cover,  $Y$  smooth Enriques surface,  $X$  smooth  $K3$  surface.

## Objects of interest

- $K3$  carpet  $\tilde{Y}$ : Cohen–Macaulay double structure on  $Y$ .
- $\tilde{Y}$  has the invariants of a smooth  $K3$  surface:

$$\omega_{\tilde{Y}} \simeq \mathcal{O}_{\tilde{Y}} \text{ and } H^1(\mathcal{O}_{\tilde{Y}}) = 0.$$



## Setup

- $X \xrightarrow{\pi} Y$ , étale double cover,  $Y$  smooth Enriques surface,  $X$  smooth  $K3$  surface.

## Objets of interest

- $K3$  carpet  $\tilde{Y}$ : Cohen–Macaulay double structure on  $Y$ .
- $\tilde{Y}$  has the invariants of a smooth  $K3$  surface:

$$\omega_{\tilde{Y}} \simeq \mathcal{O}_{\tilde{Y}} \text{ and } H^1(\mathcal{O}_{\tilde{Y}}) = 0.$$



- **Simplest** carpet  $\tilde{Y}$ : the first infinitesimal neighborhood of  $Y = \mathbf{A}^2$  in  $\mathbf{A}^3$

$$\tilde{Y} = \text{Spec } \mathbf{C}[x, y, \epsilon]/\epsilon^2.$$

- The **ideal** of  $Y = \mathbf{A}^2$  in  $\tilde{Y}$  has **square zero**. This ideal is the **free**  $\mathbf{C}[x, y]$ -module  $\mathbf{C}[x, y]\epsilon$ .
- By **gluing** trivial affine carpets we obtain a **carpet** on  $\mathbf{P}^2$ .  
The gluing isomorphism assigns to the carpet its **conormal bundle**.

### Example

There is, up to isomorphism, **one non-split** carpet on  $\mathbf{P}^2$  with conormal bundle  $\omega_{\mathbf{P}^2}$ .

**Reason:**

$$\dim \text{Ext}^1(\Omega_{\mathbf{P}^2}, \omega_{\mathbf{P}^2}) = 1.$$



- **Simplest** carpet  $\tilde{Y}$ : the first infinitesimal neighborhood of  $Y = \mathbf{A}^2$  in  $\mathbf{A}^3$

$$\tilde{Y} = \text{Spec } \mathbf{C}[x, y, \epsilon]/\epsilon^2.$$

- The **ideal** of  $Y = \mathbf{A}^2$  in  $\tilde{Y}$  has **square zero**. This ideal is the **free**  $\mathbf{C}[x, y]$ -module  $\mathbf{C}[x, y]\epsilon$ .
- By **gluing** trivial affine carpets we obtain a **carpet** on  $\mathbf{P}^2$ . The gluing isomorphism assigns to the carpet its **conormal bundle**.

### Example

There is, up to isomorphism, **one non-split** carpet on  $\mathbf{P}^2$  with conormal bundle  $\omega_{\mathbf{P}^2}$ .

**Reason:**

$$\dim \text{Ext}^1(\Omega_{\mathbf{P}^2}, \omega_{\mathbf{P}^2}) = 1.$$



- **Simplest** carpet  $\tilde{Y}$ : the first infinitesimal neighborhood of  $Y = \mathbf{A}^2$  in  $\mathbf{A}^3$

$$\tilde{Y} = \text{Spec } \mathbf{C}[x, y, \epsilon]/\epsilon^2.$$

- The **ideal** of  $Y = \mathbf{A}^2$  in  $\tilde{Y}$  has **square zero**. This ideal is the **free**  $\mathbf{C}[x, y]$ -module  $\mathbf{C}[x, y]\epsilon$ .
- By **gluing** trivial affine carpets we obtain a **carpet** on  $\mathbf{P}^2$ . The gluing isomorphism assigns to the carpet its **conormal bundle**.

### Example

There is, up to isomorphism, **one non-split** carpet on  $\mathbf{P}^2$  with conormal bundle  $\omega_{\mathbf{P}^2}$ .

**Reason:**

$$\dim \text{Ext}^1(\Omega_{\mathbf{P}^2}, \omega_{\mathbf{P}^2}) = 1.$$



## Embedded carpets: Thickening in a normal direction (Ferrand–Szipro construction)

- Any **embedded** carpet  $Y \subset \tilde{Y} \subset Z$  is obtained <sup>(1)</sup> as the “thickening” of  $Y$  in a normal direction given by a **quotient**,

$$\mathcal{I}_{Y,Z} / \mathcal{I}_{Y,Z}^2 \twoheadrightarrow \mathcal{E}.$$

- The **ideal**  $\mathcal{I}_{\tilde{Y},Z}$  of  $\tilde{Y}$  in  $Z$  is the **kernel** of the composite map

$$\mathcal{I}_{\tilde{Y},Z} = \ker(\mathcal{I}_{Y,Z} \twoheadrightarrow \mathcal{I}_{Y,Z} / \mathcal{I}_{Y,Z}^2 \twoheadrightarrow \mathcal{E}).$$

---

<sup>1</sup>Hulek, Van de Ven, *The Horrocks-Mumford bundle and the Ferrand construction*, Manuscripta Math. **50**, 313–315 (1985).

F. J. Gallego and B. P. Purnaprajna, *Degenerations of K3 surfaces in projective space*, Trans. Amer. Math. Soc. **349** (1997), no. 6, 2477–2492.



## Embedded carpets: Thickening in a normal direction (Ferrand–Sziro construction)

- Any **embedded** carpet  $Y \subset \tilde{Y} \subset Z$  is obtained <sup>(1)</sup> as the “thickening” of  $Y$  in a normal direction given by a **quotient**,

$$\mathcal{I}_{Y,Z} / \mathcal{I}_{Y,Z}^2 \twoheadrightarrow \mathcal{E}.$$

- The **ideal**  $\mathcal{I}_{\tilde{Y},Z}$  of  $\tilde{Y}$  in  $Z$  is the **kernel** of the composite map

$$\mathcal{I}_{\tilde{Y},Z} = \ker(\mathcal{I}_{Y,Z} \twoheadrightarrow \mathcal{I}_{Y,Z} / \mathcal{I}_{Y,Z}^2 \twoheadrightarrow \mathcal{E}).$$

---

<sup>1</sup>Hulek, Van de Ven, *The Horrocks-Mumford bundle and the Ferrand construction*, Manuscripta Math. **50**, 313–315 (1985).

F. J. Gallego and B. P. Purnaprajna, *Degenerations of K3 surfaces in projective space*, Trans. Amer. Math. Soc. **349** (1997), no. 6, 2477–2492.



## Formal definition

- $Y$  reduced connected scheme.  $\mathcal{E}$  line bundle on  $Y$ .

Definition <sup>(2)</sup>

A **ribbon** on  $Y$  with **conormal bundle**  $\mathcal{E}$  is a scheme  $\tilde{Y}$  such that:

- $\tilde{Y}_{\text{red}} = Y$ ,
  - $\mathcal{I}_{Y, \tilde{Y}}^2 = 0$  (so that  $\mathcal{I}_{Y, \tilde{Y}}$  is an  $\mathcal{O}_Y$ -module) and,
  - $\mathcal{I}_{Y, \tilde{Y}} \simeq \mathcal{E}$  as  $\mathcal{O}_Y$ -modules.
- When  $Y$  surface, we call  $\tilde{Y}$  a **carpet**.
  - If  $\mathcal{E}$  has arbitrary rank  $n - 1$  we call  $\tilde{Y}$  an  **$n$ -rope**.
  - **Split ribbon**: The **unique** ribbon  $\tilde{Y}$  s.t. the embedding  $Y \hookrightarrow \tilde{Y}$  admits a **retraction**  $\tilde{Y} \rightarrow Y$ .

<sup>2</sup>D. Bayer and D. Eisenbud, *Ribbons and their canonical embeddings*, Trans. Amer. Math. Soc. 347 (1995),



## Formal definition

- $Y$  reduced connected scheme.  $\mathcal{E}$  line bundle on  $Y$ .

Definition <sup>(2)</sup>

A **ribbon** on  $Y$  with **conormal bundle**  $\mathcal{E}$  is a scheme  $\tilde{Y}$  such that:

- $\tilde{Y}_{\text{red}} = Y$ ,
  - $\mathcal{I}_{Y, \tilde{Y}}^2 = 0$  (so that  $\mathcal{I}_{Y, \tilde{Y}}$  is an  $\mathcal{O}_Y$ -module) and,
  - $\mathcal{I}_{Y, \tilde{Y}} \simeq \mathcal{E}$  as  $\mathcal{O}_Y$ -modules.
- When  $Y$  surface, we call  $\tilde{Y}$  a **carpet**.
  - If  $\mathcal{E}$  has arbitrary rank  $n - 1$  we call  $\tilde{Y}$  an  $n$ -**rope**.
  - **Split ribbon**: The **unique** ribbon  $\tilde{Y}$  s.t. the embedding  $Y \hookrightarrow \tilde{Y}$  admits a **retraction**  $\tilde{Y} \rightarrow Y$ .

<sup>2</sup>D. Bayer and D. Eisenbud, *Ribbons and their canonical embeddings*, Trans. Amer. Math. Soc. **347** (1995),



## Formal definition

- $Y$  reduced connected scheme.  $\mathcal{E}$  line bundle on  $Y$ .

Definition (<sup>2</sup>)

A **ribbon** on  $Y$  with **conormal bundle**  $\mathcal{E}$  is a scheme  $\tilde{Y}$  such that:

- $\tilde{Y}_{\text{red}} = Y$ ,
  - $\mathcal{I}_{Y, \tilde{Y}}^2 = 0$  (so that  $\mathcal{I}_{Y, \tilde{Y}}$  is an  $\mathcal{O}_Y$ -module) and,
  - $\mathcal{I}_{Y, \tilde{Y}} \simeq \mathcal{E}$  as  $\mathcal{O}_Y$ -modules.
- When  $Y$  surface, we call  $\tilde{Y}$  a **carpet**.
  - If  $\mathcal{E}$  has arbitrary rank  $n - 1$  we call  $\tilde{Y}$  an  $n$ -**rope**.
  - **Split ribbon**: The **unique** ribbon  $\tilde{Y}$  s.t. the embedding  $Y \hookrightarrow \tilde{Y}$  admits a **retraction**  $\tilde{Y} \rightarrow Y$ .

<sup>2</sup>D. Bayer and D. Eisenbud, *Ribbons and their canonical embeddings*, Trans. Amer. Math. Soc. **347** (1995),



Restricted cotangent sequence <sup>(3)</sup>

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O}_{\tilde{Y}} & \longrightarrow & \mathcal{O}_Y \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \Omega_{\tilde{Y}}|_Y & \longrightarrow & \Omega_Y \longrightarrow 0
 \end{array}$$

$$\{\text{Ropes conormal } \mathcal{E}\} / \simeq \longleftrightarrow \text{Ext}^1(\Omega_Y, \mathcal{E}) / \text{Aut } \mathcal{E}$$

( $\simeq$  inducing identity on  $Y$ )

<sup>3</sup> D. Bayer and D. Eisenbud, *Ribbons and their canonical embeddings*, Trans. Amer. Math. Soc. **347** (1995), no. 3, 719–756.



## Part II

# $K3$ carpets



- $Y$  smooth regular surface, i.e.  $H^1(\mathcal{O}_Y) = 0$ .

### Definition

A  $K3$  carpet  $\tilde{Y}$  is a carpet on  $Y$  such that  $\omega_{\tilde{Y}} \simeq \mathcal{O}_{\tilde{Y}}$ .

- $\tilde{Y}$  proper over  $\mathbf{C} \Rightarrow$  there exists  $\omega_{\tilde{Y}}$ , Grothendieck dualizing sheaf on  $\tilde{Y}$  with nice functorial properties. <sup>(4)</sup>
- $\tilde{Y}$  locally Gorenstein  $\Rightarrow \omega_{\tilde{Y}}$  invertible. <sup>(5)</sup>

<sup>4</sup> Kleiman, *Relative duality for quasicohherent sheaves*, *Compositio Math.* 41 (1980), no. 1, 39–60.

<sup>5</sup> Hartshorne, *Residues and duality*, L.N.M. 20.

Conrad, *Grothendieck duality and base change*, L.N.M. 1750.



- $Y$  smooth regular surface, i.e.  $H^1(\mathcal{O}_Y) = 0$ .

## Definition

A K3 carpet  $\tilde{Y}$  is a carpet on  $Y$  such that  $\omega_{\tilde{Y}} \simeq \mathcal{O}_{\tilde{Y}}$ .

- $\tilde{Y}$  proper over  $\mathbf{C} \Rightarrow$  there exists  $\omega_{\tilde{Y}}$ , Grothendieck **dualizing sheaf** on  $\tilde{Y}$  with nice functorial properties. <sup>(4)</sup>
- $\tilde{Y}$  locally Gorenstein  $\Rightarrow \omega_{\tilde{Y}}$  **invertible**. <sup>(5)</sup>

<sup>4</sup> Kleiman, *Relative duality for quasicoherent sheaves*, Compositio Math. **41** (1980), no. 1, 39–60.

<sup>5</sup> Hartshorne, *Residues and duality*, L.N.M. 20.

Conrad, *Grothendieck duality and base change*, L.N.M. 1750.



- $Y$  smooth regular surface, i.e.  $H^1(\mathcal{O}_Y) = 0$ .

### Definition

A K3 carpet  $\tilde{Y}$  is a carpet on  $Y$  such that  $\omega_{\tilde{Y}} \simeq \mathcal{O}_{\tilde{Y}}$ .

- $\tilde{Y}$  proper over  $\mathbf{C} \Rightarrow$  there exists  $\omega_{\tilde{Y}}$ , Grothendieck **dualizing sheaf** on  $\tilde{Y}$  with nice functorial properties. <sup>(4)</sup>
- $\tilde{Y}$  locally Gorenstein  $\Rightarrow \omega_{\tilde{Y}}$  **invertible**. <sup>(5)</sup>

<sup>4</sup> Kleiman, *Relative duality for quasicohherent sheaves*, Compositio Math. **41** (1980), no. 1, 39–60.

<sup>5</sup> Hartshorne, *Residues and duality*, L.N.M. 20.

Conrad, *Grothendieck duality and base change*, L.N.M. 1750.



## Characterization of $K3$ carpets

- $\tilde{Y}$  is a  $K3$  carpet *iff*  $\mathcal{E} \simeq \omega_Y$ .

- Let  $\tilde{Y}$  be a  $K3$  carpet. Then  $H^1(\mathcal{O}_{\tilde{Y}}) = 0$ .

- $\{ \text{Non-split } K3 \text{ carpets} \} / \sim \longleftrightarrow \mathbf{P}(\text{Ext}_Y^1(\Omega_Y, \omega_Y))$ .

- If  $Y$  smooth Enriques surface, i.e. regular and  $\omega_Y^2 = \mathcal{O}_Y$ , then:

- ★  $\dim \text{Ext}_Y^1(\Omega_Y, \omega_Y) = h^{1,1} = 10$ .



## Characterization of $K3$ carpets

•  $\tilde{Y}$  is a  $K3$  carpet *iff*  $\mathcal{E} \simeq \omega_Y$ .

• Let  $\tilde{Y}$  be a  $K3$  carpet. Then  $H^1(\mathcal{O}_{\tilde{Y}}) = 0$ .

• { Non-split  $K3$  carpets } /  $\sim \longleftrightarrow \mathbf{P}(\text{Ext}_Y^1(\Omega_Y, \omega_Y))$ .

• If  $Y$  smooth **Enriques** surface, i.e. regular and  $\omega_Y^2 = \mathcal{O}_Y$ , then:

★  $\dim \text{Ext}_Y^1(\Omega_Y, \omega_Y) = h^{1,1} = 10$ .



## Characterization of K3 carpets

- $\tilde{Y}$  is a K3 carpet *iff*  $\mathcal{E} \simeq \omega_Y$ .
- Let  $\tilde{Y}$  be a K3 carpet. Then  $H^1(\mathcal{O}_{\tilde{Y}}) = 0$ .
- { Non-split K3 carpets } /  $\sim \longleftrightarrow \mathbf{P}(\text{Ext}_Y^1(\Omega_Y, \omega_Y))$ .
- If  $Y$  smooth **Enriques** surface, i.e. regular and  $\omega_Y^2 = \mathcal{O}_Y$ , then:
  - ★  $\dim \text{Ext}_Y^1(\Omega_Y, \omega_Y) = h^{1,1} = 10$ .



## Characterization of $K3$ carpets

- $\tilde{Y}$  is a  $K3$  carpet *iff*  $\mathcal{E} \simeq \omega_Y$ .

- Let  $\tilde{Y}$  be a  $K3$  carpet. Then  $H^1(\mathcal{O}_{\tilde{Y}}) = 0$ .

- $\{ \text{Non-split } K3 \text{ carpets} \} / \sim \longleftrightarrow \mathbf{P}(\text{Ext}_Y^1(\Omega_Y, \omega_Y))$ .

- If  $Y$  smooth **Enriques** surface, i.e. regular and  $\omega_Y^2 = \mathcal{O}_Y$ , then:

- ★  $\dim \text{Ext}_Y^1(\Omega_Y, \omega_Y) = h^{1,1} = 10$ .



## Setup

### Setup

- $Y$  smooth Enriques surface:  $\omega_Y^2 = \mathcal{O}_Y$ ,  $H^1(\mathcal{O}_Y) = 0$ .
- $X \xrightarrow{\pi} Y$ , étale **double cover**.
  - $X$  smooth  $K3$  surface:  $\omega_X = \mathcal{O}_X$ ,  $H^1(\mathcal{O}_X) = 0$ .
  - $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \omega_Y$ .
  - $\omega_Y$  **trace zero module** of  $\pi$  = **conormal bundle** of  $K3$  carpets.

“Moral” consequence:

- $K3$  carpets on Enriques surfaces, must be **smoothable** (like  $K3$  carpets on rational normal scrolls <sup>(6)</sup>).

---

<sup>6</sup> Gallego and Purnaprajna, *Degenerations of  $K3$  surfaces in projective space*, Trans. Amer. Math. Soc. 349 (1997), no. 6, 2477–2492.



## Setup

### Setup

- $Y$  smooth Enriques surface:  $\omega_Y^2 = \mathcal{O}_Y$ ,  $H^1(\mathcal{O}_Y) = 0$ .
- $X \xrightarrow{\pi} Y$ , étale **double cover**.
  - $X$  smooth  $K3$  surface:  $\omega_X = \mathcal{O}_X$ ,  $H^1(\mathcal{O}_X) = 0$ .
  - $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \omega_Y$ .
  - $\omega_Y$  **trace zero module** of  $\pi$  = **conormal bundle** of  $K3$  carpets.

### “Moral” consequence:

- $K3$  carpets on Enriques surfaces, must be **smoothable** (like  $K3$  carpets on rational normal scrolls <sup>(6)</sup>).

---

<sup>6</sup> Gallego and Purnaprajna, *Degenerations of  $K3$  surfaces in projective space*, Trans. Amer. Math. Soc. **349** (1997), no. 6, 2477–2492.



## Antecedents

- Double covers and double structures

- ★ Lung-Ying Fong, *Rational ribbons and deformation of hyperelliptic curves*, J. Algebraic Geom. **2** (1993), no. 2, 295–307.

- ★ Ribbons on  $\mathbf{P}^1$  and hyperelliptic canonical morphisms.

- ★ Francisco J. Gallego and Bangere P. Purnaprajna, *Degenerations of K3 surfaces in projective space*, Trans. Amer. Math. Soc. **349** (1997), no. 6, 2477–2492.

- ★ K3 carpets on rational normal scrolls and hyperelliptic smooth K3 surfaces.

- Finite covers and ropes on curves

- ★ M. González, *Smoothing of ribbons over curves*, J. reine angew. Math. **591** (2006), 201–235.

- ★ General infinitesimal theory relating first-order infinitesimal deformations of finite morphisms to ropes.

- ★ F.J. Gallego, M. González, and B.P. Purnaprajna, *Deformation of finite morphisms and smoothing of ropes*, arXiv:math.AG/0502467.

- ★ Smoothing of ropes of any multiplicity on curves with applications to ropes of multiplicity 3 on  $\mathbf{P}^1$



## Antecedents

- **Double covers and double structures**

- ★ Lung-Ying Fong, *Rational ribbons and deformation of hyperelliptic curves*, J. Algebraic Geom. **2** (1993), no. 2, 295–307.

- ★ Ribbons on  $\mathbf{P}^1$  and hyperelliptic canonical morphisms.

- ★ Francisco J. Gallego and Bangere P. Purnaprajna, *Degenerations of K3 surfaces in projective space*, Trans. Amer. Math. Soc. **349** (1997), no. 6, 2477–2492.

- ★ K3 carpets on rational normal scrolls and hyperelliptic smooth K3 surfaces.

- **Finite covers and ropes on curves**

- ★ M. González, *Smoothing of ribbons over curves*, J. reine angew. Math. **591** (2006), 201–235.

- ★ General infinitesimal theory relating first-order infinitesimal deformations of finite morphisms to ropes.

- ★ F.J. Gallego, M. González, and B.P. Purnaprajna, *Deformation of finite morphisms and smoothing of ropes*, arXiv:math.AG/0502467.

- ★ Smoothing of ropes of any multiplicity on curves with applications to ropes of multiplicity 3 on  $\mathbf{P}^1$ .



## Definition

- A **smoothing** for  $\tilde{Y}$  is a flat proper integral family  $\mathcal{Y}$  over a smooth affine curve  $T$ , s. t.
  - for  $0 \in T$ ,  $\mathcal{Y}_0 = \tilde{Y}$  and,
  - for  $0 \neq t \in T$ ,  $\mathcal{Y}_t$  is a smooth, irreducible, and, in our case, projective  $K3$  surface.
- **Embedded smoothing**: Same as above, but
  - Ambient variety  $Z$ ,  $Y \subset \tilde{Y} \subset Z$  (closed subschemes) and,
  - $\mathcal{Y} \subset Z \times T$  (closed subscheme).



## Definition

- A **smoothing** for  $\tilde{Y}$  is a flat proper integral family  $\mathcal{Y}$  over a smooth affine curve  $T$ , s. t.
  - for  $0 \in T$ ,  $\mathcal{Y}_0 = \tilde{Y}$  and,
  - for  $0 \neq t \in T$ ,  $\mathcal{Y}_t$  is a smooth, irreducible, and, in our case, projective  $K3$  surface.
- **Embedded** smoothing: Same as above, but
  - Ambient variety  $Z$ ,  $Y \subset \tilde{Y} \subset Z$  (closed subschemes) and,
  - $\mathcal{Y} \subset Z \times T$  (closed subscheme).



## Target result

### Theorem

*Any projective K3 carpet  $\tilde{Y}$  on an Enriques  $Y$  is smoothable.*



## Target result

### Theorem

Let  $\tilde{Y}$  *corresponding* to  $\tau \in \text{Ext}^1(\Omega_Y, \omega_Y)$ .

- $\tilde{Y}$  is *projective* iff there exists an *ample* divisor  $D$  on  $Y$  s.t.  $\int_D \tau = 0$ , where  $\tau \in H^{1,1}(Y) = H^2(Y, \mathbf{C})$ .
- *Projective K3 carpets on  $Y$  are parametrized by countably infinitely many hyperplanes of the 9-dimensional space  $\mathbf{P}(\text{Ext}^1(\Omega_Y, \omega_Y))$ , corresponding to the classes in  $NS(Y)$  of primitive ample divisors on  $Y$ .*



## Target result

### Theorem

Let  $\tilde{Y}$  *corresponding* to  $\tau \in \text{Ext}^1(\Omega_Y, \omega_Y)$ .

- $\tilde{Y}$  is *projective* iff there exists an *ample* divisor  $D$  on  $Y$  s.t.  $\int_D \tau = 0$ , where  $\tau \in H^{1,1}(Y) = H^2(Y, \mathbf{C})$ .
- Projective K3 carpets on  $Y$  are parametrized by *countably infinitely many hyperplanes* of the 9-dimensional space  $\mathbf{P}(\text{Ext}^1(\Omega_Y, \omega_Y))$ , corresponding to the classes in  $NS(Y)$  of primitive ample divisors on  $Y$ .



## Target result

### Theorem

Let  $\tilde{Y}$  *corresponding* to  $\tau \in \text{Ext}^1(\Omega_Y, \omega_Y)$ .

- $\tilde{Y}$  is *projective* iff there exists an *ample* divisor  $D$  on  $Y$  s.t.  $\int_D \tau = 0$ , where  $\tau \in H^{1,1}(Y) = H^2(Y, \mathbf{C})$ .
- Projective K3 carpets on  $Y$  are parametrized by *countably infinitely many hyperplanes* of the 9-dimensional space  $\mathbf{P}(\text{Ext}^1(\Omega_Y, \omega_Y))$ , corresponding to the classes in  $\text{NS}(Y)$  of primitive ample divisors on  $Y$ .

Let  $Y \hookrightarrow \mathbf{P}^N$  with *sectional genus*  $g$ .

- Embedded K3 carpets in  $\mathbf{P}^N$  supported on  $Y$  are parametrized by a *non-empty open set* in

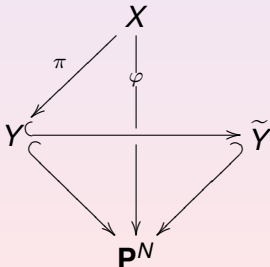
$$\mathbf{P}(H^0(\mathcal{N}_{Y, \mathbf{P}^N} \otimes \omega_Y)),$$

whose dimension is  $g(N+1) + 8$ .



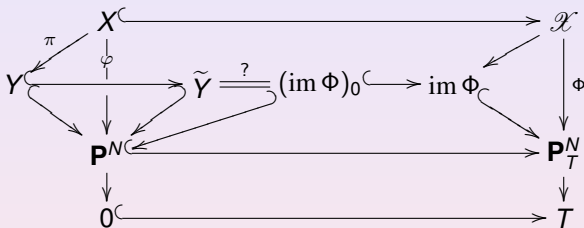
## Strategy

- $Y \subset \tilde{Y} \subset \mathbf{P}^N$ ,  $\tilde{Y}$  a K3 carpet.
- $X \xrightarrow{\pi} Y$ , étale double covering s.t.  $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \omega_Y$ ,
- $X$  smooth K3 surface.





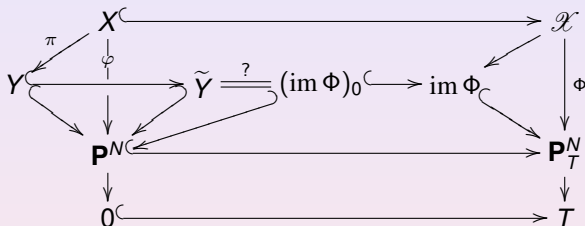
## Strategy



- $\mathcal{X}_0 = X$ ,  $\Phi_0 = \varphi$ ,  $\Phi_t$  **embedding** of a smooth K3 surface  $\mathcal{X}_t$ ,  $t \neq 0$ .
- $\text{im}(\Phi_t) = (\text{im } \Phi)_t$ ,  $t \neq 0$  smooth K3 surface.
- $Y = \text{im}(\Phi_0) \subsetneq (\text{im } \Phi)_0$ .
- $(\text{im } \Phi)_0$  has same Hilbert polynomial as an embedded K3 carpet on  $Y$ .  
 (reason: trace zero module of  $\pi =$  conormal bundle of K3 carpets)
- So, if  $\tilde{Y} \subset (\text{im } \Phi)_0$  then  $\tilde{Y} = (\text{im } \Phi)_0$ .



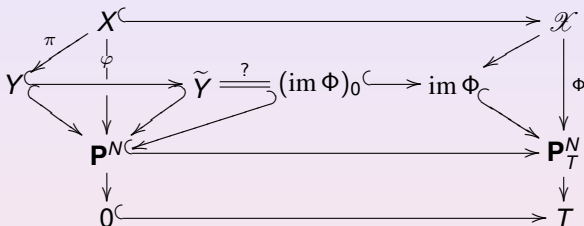
## Strategy



- $\mathcal{X}_0 = X$ ,  $\Phi_0 = \varphi$ ,  $\Phi_t$  **embedding** of a smooth K3 surface  $\mathcal{X}_t$ ,  $t \neq 0$ .
- $\text{im}(\Phi_t) = (\text{im } \Phi)_t$ ,  $t \neq 0$  smooth K3 surface.
- $Y = \text{im}(\Phi_0) \subsetneq (\text{im } \Phi)_0$ .
- $(\text{im } \Phi)_0$  has same Hilbert polynomial as an embedded K3 carpet on  $Y$ .  
(reason: trace zero module of  $\pi =$  conormal bundle of K3 carpets)
- So, if  $\tilde{Y} \subset (\text{im } \Phi)_0$  then  $\tilde{Y} = (\text{im } \Phi)_0$ .



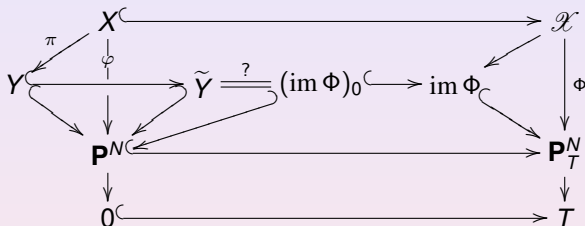
## Strategy



- $\mathcal{X}_0 = X$ ,  $\Phi_0 = \varphi$ ,  $\Phi_t$  **embedding** of a smooth K3 surface  $\mathcal{X}_t$ ,  $t \neq 0$ .
- $\text{im}(\Phi_t) = (\text{im} \Phi)_t$ ,  $t \neq 0$  smooth K3 surface.
- $Y = \text{im}(\Phi_0) \subsetneq (\text{im} \Phi)_0$ .
- $(\text{im} \Phi)_0$  has same Hilbert polynomial as an embedded K3 carpet on  $Y$ .  
(reason: **trace zero module** of  $\pi =$  **conormal bundle** of K3 carpets)
- So, if  $\tilde{Y} \subset (\text{im} \Phi)_0$  then  $\tilde{Y} = (\text{im} \Phi)_0$ .



## Strategy



- $\mathcal{X}_0 = X$ ,  $\Phi_0 = \varphi$ ,  $\Phi_t$  **embedding** of a smooth K3 surface  $\mathcal{X}_t$ ,  $t \neq 0$ .
- $\text{im}(\Phi_t) = (\text{im} \Phi)_t$ ,  $t \neq 0$  smooth K3 surface.
- $Y = \text{im}(\Phi_0) \subsetneq (\text{im} \Phi)_0$ .
- $(\text{im} \Phi)_0$  has same Hilbert polynomial as an embedded K3 carpet on  $Y$ .  
(reason: **trace zero module** of  $\pi =$  **conormal bundle** of K3 carpets)
- So, if  $\tilde{Y} \subset (\text{im} \Phi)_0$  then  $\tilde{Y} = (\text{im} \Phi)_0$ .



## Embedded infinitesimal smoothing

- First-order infinitesimal deformation of  $X \xrightarrow{\varphi} \mathbf{P}^N$ :

$$\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}_{\Delta}^N, \quad (\Delta = \text{Spec } k[\epsilon]/\epsilon^2)$$

such that

$$(\text{im } \tilde{\varphi})_0 = \tilde{Y}.$$



## Theorem

Let  $\tilde{Y} \subset \mathbf{P}^N$  be a projective K3 carpet. There exists a first-order infinitesimal deformation

$$\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}^N \times \Delta$$

of  $X \xrightarrow{\varphi} \mathbf{P}^N$  such that

$$(\mathrm{im} \tilde{\varphi})_0 = \tilde{Y}.$$

- Since  $\pi$  is étale, this follows from the general infinitesimal theory in (7).

---

<sup>7</sup>M. González, *Smoothing of ribbons over curves*, J. reine angew. Math. **591** (2006), 201–235.



## Outline of the infinitesimal theory

- **Normal sheaf** of  $\varphi$  defined by

$$0 \rightarrow \mathcal{T}_X \rightarrow \varphi^* \mathcal{T}_{\mathbf{P}^N} \rightarrow \mathcal{N}_\varphi \rightarrow 0$$

Nice exact sequence

$$0 \rightarrow \mathcal{N}_\pi \rightarrow \mathcal{N}_\varphi \rightarrow \pi^* \mathcal{N}_{Y, \mathbf{P}^N} \rightarrow 0$$

Key diagram

$$\begin{array}{ccc}
 H^0(\mathcal{N}_\varphi) & \xrightarrow{\delta_1} & \text{Ext}^1(\Omega_X, \mathcal{O}_X) \\
 \Phi_1 \oplus \Phi_2 \downarrow & & \downarrow \Psi_1 \oplus \Psi_2 \\
 \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \oplus \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) & \xrightarrow{\delta_3} & \text{Ext}^1(\Omega_Y, \mathcal{O}_Y) \oplus \text{Ext}^1(\Omega_Y, \mathcal{E})
 \end{array}$$



## Outline of the infinitesimal theory

- **Normal sheaf** of  $\varphi$  defined by

$$0 \rightarrow \mathcal{I}_X \rightarrow \varphi^* \mathcal{I}_{\mathbf{P}^N} \rightarrow \mathcal{N}_\varphi \rightarrow 0$$

$$\mathcal{N}_\varphi \xrightarrow{\sim} \pi^* \mathcal{N}_{Y, \mathbf{P}^N} \quad (\mathcal{N}_\pi = 0)$$

### Key diagram

$$\begin{array}{ccc}
 H^0(\mathcal{N}_\varphi) & \xrightarrow{\delta_1} & \text{Ext}^1(\Omega_X, \mathcal{O}_X) \\
 \Phi_1 \oplus \Phi_2 \parallel & & \downarrow \Psi_1 \oplus \Psi_2 \\
 \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \oplus \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) & \xrightarrow{\delta_3} & \text{Ext}^1(\Omega_Y, \mathcal{O}_Y) \oplus \text{Ext}^1(\Omega_Y, \mathcal{E})
 \end{array}$$



## Outline of the general infinitesimal theory

- $X \xrightarrow{\pi} Y \hookrightarrow Z$ ,  $X \xrightarrow{\varphi} Z$ ,  $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$ .

$$\begin{array}{ccc}
 H^0(\mathcal{N}_\varphi) & \xrightarrow{\delta_1} & \text{Ext}^1(\Omega_X, \mathcal{O}_X) \\
 \downarrow \Phi_1 \oplus \Phi_2 & & \downarrow \Psi_1 \oplus \Psi_2 \\
 \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \oplus \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) & \xrightarrow{\delta_3} & \text{Ext}^1(\Omega_Y, \mathcal{O}_Y) \oplus \text{Ext}^1(\Omega_Y, \mathcal{E}).
 \end{array}$$

- $H^0(\mathcal{N}_\varphi) \longleftrightarrow \{ \text{1st-order } \infty\text{-mal deformations } (\tilde{X}, \tilde{\varphi}) \text{ of } (X, \varphi) \}$ .
- $\text{Ext}^1(\Omega_X, \mathcal{O}_X) \longleftrightarrow \{ \tilde{X} \}$ .
- $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) \longleftrightarrow \{ \text{pairs } (\tilde{Y}, \tilde{i}), \tilde{Y} \text{ rope conormal } \mathcal{E} \text{ and } \tilde{Y} \xrightarrow{\tilde{i}} Z \text{ extension morphism of } Y \hookrightarrow Z \}$ .
- $\text{Ext}^1(\Omega_Y, \mathcal{E}) \longleftrightarrow \{ \tilde{Y} \}$ .
- $\text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \longleftrightarrow \{ \text{1st-order deformations } \tilde{Y} \subset Z \times \Delta \text{ of } Y \subset Z \}$ .
- $\text{Ext}^1(\Omega_Y, \mathcal{O}_Y) \longleftrightarrow \{ \tilde{Y} \}$ .



## Outline of the general infinitesimal theory

$$\begin{array}{ccc}
 H^0(\mathcal{N}_\varphi) & \xrightarrow{\delta_1} & \text{Ext}^1(\Omega_X, \mathcal{O}_X) \\
 \downarrow \Phi_1 \oplus \Phi_2 & & \downarrow \Psi_1 \oplus \Psi_2 \\
 \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \oplus \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) & \xrightarrow{\delta_3} & \text{Ext}^1(\Omega_Y, \mathcal{O}_Y) \oplus \text{Ext}^1(\Omega_Y, \mathcal{E}).
 \end{array}$$

Let  $(\tilde{X}, \tilde{\varphi}) = \nu \in H^0(\mathcal{N}_\varphi)$  and  $(\tilde{Y}, \tilde{i}) = \Phi_2 \nu \in \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E})$ .

• Then:<sup>8</sup>  $(\text{im } \tilde{\varphi})_0 = \text{im}(\tilde{Y} \xrightarrow{\tilde{i}} Z)$ .

<sup>8</sup>M. González, *Smoothing...*, Crelle **591** (2006)



## Smoothing of any first-order infinitesimal deformation of $\varphi$

### Theorem

Let  $Y \hookrightarrow \mathbf{P}^N$  with sectional genus  $g$  and  $N \leq 2g - 1$ . Let  $\varphi : X \xrightarrow{\pi} Y \hookrightarrow \mathbf{P}^N$ . For *any* first-order infinitesimal deformation  $\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}_{\Delta}^N$  of  $\varphi$ , there exists a smooth proper family  $\mathcal{X} \rightarrow T$  and  $\mathcal{X} \xrightarrow{\Phi} \mathbf{P}_T^N$ , over a smooth affine curve  $(T, 0)$ , s.t.

- the general fiber  $\mathcal{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N$ ,  $t \in T - 0$ , is a *closed immersion* of a smooth K3 surface; and
- the fiber  $\mathcal{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}_{\Delta}^N$  over the *tangent* vector at  $0 \in T$  is  $\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}_{\Delta}^N$ .



## Embedded smoothing

### Theorem

Let  $\tilde{Y} \subset \mathbf{P}^N$  projective K3 carpet, on  $Y \subset \mathbf{P}^N$  with  $N \leq 2g - 1$ .

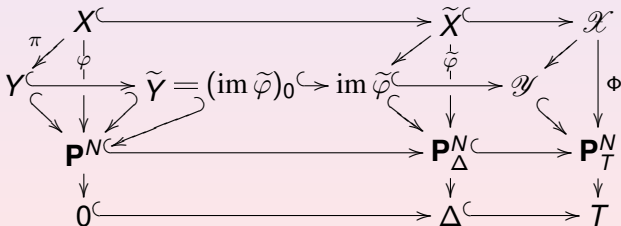
There exists  $\mathcal{X} \xrightarrow{\Phi} \mathbf{P}_T^N$  as above, s.t. for  $\mathcal{Y} := \text{im } \Phi$

- the general fiber  $\mathcal{Y}_t$ ,  $t \in T - 0$ , is a **smooth** non-degenerate K3 surface in  $\mathbf{P}^N$ ,
- the central fiber  $\mathcal{Y}_0 \subset \mathbf{P}^N$  is  $\tilde{Y} \subset \mathbf{P}^N$ .

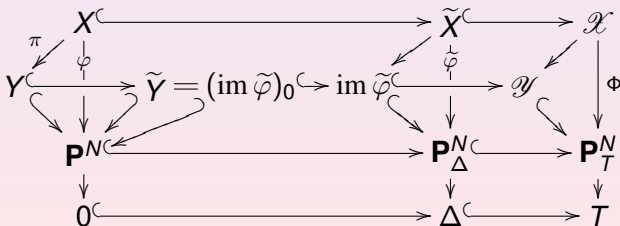


► Skip

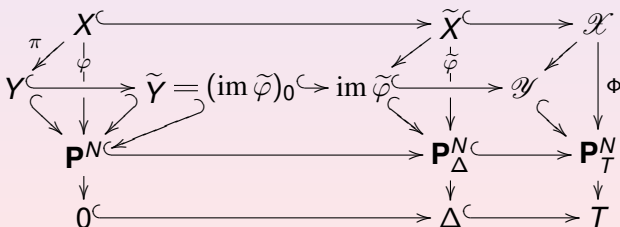
- Let  $Y \subset \tilde{Y} \subset \mathbf{P}^N$  and  $\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}_{\Delta}^N$  s.t.  $\tilde{Y} = (\text{im } \tilde{\varphi})_0$ .
- Let  $\mathcal{X} \xrightarrow{\Phi} \mathbf{P}_T^N$ , s.t.
  - $\mathcal{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N, t \neq 0$  embedding of a smooth  $K3$  surface,
  - $(\mathcal{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}_{\Delta}^N) = (\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}_{\Delta}^N), (\mathcal{X}_0 \xrightarrow{\Phi_0} \mathbf{P}^N) = (X \xrightarrow{\varphi} \mathbf{P}^N)$ .



- Let  $Y \subset \tilde{Y} \subset \mathbf{P}^N$  and  $\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}_{\Delta}^N$  s.t.  $\tilde{Y} = (\text{im } \tilde{\varphi})_0$ .
- Let  $\mathcal{X} \xrightarrow{\Phi} \mathbf{P}_T^N$ , s.t.
  - $\mathcal{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N, t \neq 0$  embedding of a smooth K3 surface,
  - $(\mathcal{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}_{\Delta}^N) = (\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}_{\Delta}^N), (\mathcal{X}_0 \xrightarrow{\Phi_0} \mathbf{P}^N) = (X \xrightarrow{\varphi} \mathbf{P}^N)$ .
- Then,  $\mathcal{Y} := \text{im } \Phi$ , is  $T$ -flat,
  - $\mathcal{Y}_t, t \neq 0$  is a smooth K3 surface, and

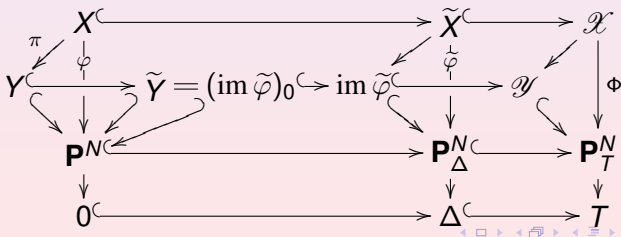


- Let  $\mathcal{X} \xrightarrow{\Phi} \mathbf{P}_T^N$ , s.t.
  - $\mathcal{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N, t \neq 0$  embedding of a smooth K3 surface,
  - $(\mathcal{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}_\Delta^N) = (\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}_\Delta^N), (\mathcal{X}_0 \xrightarrow{\Phi_0} \mathbf{P}^N) = (X \xrightarrow{\varphi} \mathbf{P}^N)$ .
- Then,  $\mathcal{Y} := \text{im } \Phi$ , is  $T$ -flat,
  - $\mathcal{Y}_t, t \neq 0$  is a smooth K3 surface, and
  - $\tilde{Y} = (\text{im } \tilde{\varphi})_0 \subset \mathcal{Y}_0$ .

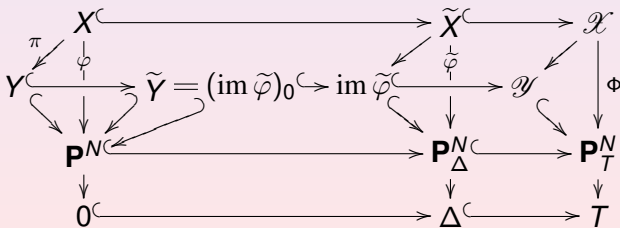


- Let  $\mathcal{X} \xrightarrow{\Phi} \mathbf{P}_T^N$ , s.t.
  - $\mathcal{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N, t \neq 0$  embedding of a smooth K3 surface,
  - $(\mathcal{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}_\Delta^N) = (\tilde{\mathcal{X}} \xrightarrow{\tilde{\Phi}} \mathbf{P}_\Delta^N), (\mathcal{X}_0 \xrightarrow{\Phi_0} \mathbf{P}^N) = (X \xrightarrow{\varphi} \mathbf{P}^N).$
- Then,  $\mathcal{Y} := \text{im } \Phi$ , is  $T$ -flat,
  - $\mathcal{Y}_t, t \neq 0$  is a smooth K3 surface, and
  - $\tilde{\mathcal{Y}} = (\text{im } \tilde{\varphi})_0 \subset \mathcal{Y}_0$ , and
  - trace zero module** of  $\pi =$  **conormal bundle** of K3 carpets.

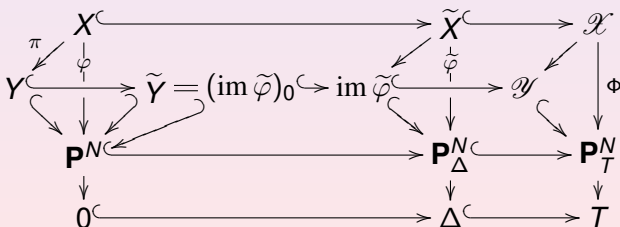
Then



- Let  $\mathcal{X} \xrightarrow{\phi} \mathbf{P}_T^N$ , s.t.
  - $\mathcal{X}_t \xrightarrow{\phi_t} \mathbf{P}^N, t \neq 0$  embedding of a smooth K3 surface,
  - $(\mathcal{X}_\epsilon \xrightarrow{\phi_\epsilon} \mathbf{P}_\Delta^N) = (\tilde{\mathcal{X}} \xrightarrow{\tilde{\varphi}} \mathbf{P}_\Delta^N), (\mathcal{X}_0 \xrightarrow{\phi_0} \mathbf{P}^N) = (X \xrightarrow{\varphi} \mathbf{P}^N).$
- Then,  $\mathcal{Y} := \text{im } \phi$ , is  $T$ -flat,
  - $\mathcal{Y}_t, t \neq 0$  is a smooth K3 surface, and
  - $\tilde{Y} = (\text{im } \tilde{\varphi})_0 \subset \mathcal{Y}_0$ , and
  - $(\text{im } \tilde{\varphi})_0$  and  $\mathcal{Y}_0$  have **same Hilbert polynomial**. So



- Let  $\mathcal{X} \xrightarrow{\Phi} \mathbf{P}_T^N$ , s.t.
  - $\mathcal{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N, t \neq 0$  embedding of a smooth K3 surface,
  - $(\mathcal{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}_\Delta^N) = (\tilde{X} \xrightarrow{\tilde{\varphi}} \mathbf{P}_\Delta^N), (\mathcal{X}_0 \xrightarrow{\Phi_0} \mathbf{P}^N) = (X \xrightarrow{\varphi} \mathbf{P}^N).$
- Then,  $\mathcal{Y} := \text{im } \Phi$ , is  $T$ -flat,
  - $\mathcal{Y}_t, t \neq 0$  is a smooth K3 surface, and
  - $\tilde{Y} = (\text{im } \tilde{\varphi})_0 = \mathcal{Y}_0.$



## The Hilbert point of an embedded $K3$ carpet

### Theorem

Let  $\tilde{Y} \subset \mathbf{P}^N$  be a projective  $K3$  carpet embedded as above ( $N \leq 2g - 1$ ). Then the *Hilbert point* of  $\tilde{Y}$  is *nonsingular*.

