# Smoothing of $K3$ carpets on Enriques surfaces

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joint work with
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# Part I

## Introduction

# Setup

- $X \xrightarrow{\pi} Y$, étale double cover, $Y$ smooth Enriques surface, $X$ smooth $K3$ surface.

## Objetcs of interest

- $K3$ carpet $\widetilde{Y}$: Cohen–Macaulay double structure on $Y$.
- $\widetilde{Y}$ has the invariants of a smooth $K3$ surface:

$$\omega_{\widetilde{Y}} \simeq \mathcal{O}_{\widetilde{Y}} \quad \text{and} \quad H^1(\mathcal{O}_{\widetilde{Y}}) = 0.$$

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- Simplest carpet $\widetilde{Y}$: the first infinitesimal neighborhood of $Y = \mathbf{A}^2$ in $\mathbf{A}^3$
$$\widetilde{Y} = \operatorname{Spec} \mathbf{C}[x, y, \epsilon]/\epsilon^2.$$
- The ideal of $Y = \mathbf{A}^2$ in $\widetilde{Y}$ has square zero. This ideal is the free $\mathbf{C}[x, y]$–module $\mathbf{C}[x, y]\epsilon$.
- By gluing trivial affine carpets we obtain a carpet on $\mathbf{P}^2$. The gluing isomorphism assigns to the carpet its conormal bundle.

**Example**

There is, up to isomorphism, one non–split carpet on $\mathbf{P}^2$ with conormal bundle $\omega_{\mathbf{P}^2}$.

Reason:
$$\dim \operatorname{Ext}^1(\Omega_{\mathbf{P}^2}, \omega_{\mathbf{P}^2}) = 1.$$

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## Embedded carpets: Thickening in a normal direction (Ferrand–Szpiro construction)

- Any embedded carpet $Y \subset \widetilde{Y} \subset Z$ is obtained ([1]) as the “thickening” of $Y$ in a normal direction given by a quotient,

$$\mathscr{I}_{Y,Z}/\mathscr{I}_{Y,Z}^{2} \twoheadrightarrow \mathscr{E}.$$

- The ideal $\mathscr{I}_{\widetilde{Y},Z}$ of $\widetilde{Y}$ in $Z$ is the kernel of the composite map

$$\mathscr{I}_{\widetilde{Y},Z} = \ker\left(\mathscr{I}_{Y,Z} \twoheadrightarrow \mathscr{I}_{Y,Z}/\mathscr{I}_{Y,Z}^{2} \twoheadrightarrow \mathscr{E}\right).$$

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[1] Hulek, Van de Ven, *The Horrocks-Mumford bundle and the Ferrand construction*, Manuscripta Math. **50**, 313–315 (1985).

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# Formal definition

- $Y$ reduced connected scheme. $\mathscr{E}$ line bundle on $Y$.

**Definition ([2])**

A ribbon on $Y$ with conormal bundle $\mathscr{E}$ is a scheme $\widetilde{Y}$ such that:

- $\widetilde{Y}_{\mathrm{red}} = Y$,
- $\mathscr{I}^2_{Y,\widetilde{Y}} = 0$ (so that $\mathscr{I}_{Y,\widetilde{Y}}$ is an $\mathscr{O}_Y$–module) and,
- $\mathscr{I}_{Y,\widetilde{Y}} \simeq \mathscr{E}$ as $\mathscr{O}_Y$–modules.

- When $Y$ surface, we call $\widetilde{Y}$ a carpet.
- If $\mathscr{E}$ has arbitrary rank $n-1$ we call $\widetilde{Y}$ an $n$–rope.
- Split ribbon: The unique ribbon $\widetilde{Y}$ s.t. the embedding $Y \hookrightarrow \widetilde{Y}$ admits a retraction $\widetilde{Y} \to Y$.

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[2] D. Bayer and D. Eisenbud, *Ribbons and their canonical embeddings*, Trans. Amer. Math. Soc. **347** (1995), no. 3, 719–756.

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## Restricted cotangent sequence ([3])

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E} \arrow[r] \arrow[d, equal] & \mathscr{O}_{\tilde{Y}} \arrow[r] \arrow[d] & \mathscr{O}_Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{E} \arrow[r] & \Omega_{\tilde{Y}}|_Y \arrow[r] & \Omega_Y \arrow[r] & 0
\end{tikzcd}$$

$\{$Ropes conormal $\mathscr{E}\}/\simeq \;\longleftrightarrow\; \mathrm{Ext}^1(\Omega_Y, \mathscr{E})/\mathrm{Aut}\,\mathscr{E}$

($\simeq$ inducing identity on $Y$)

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# Part II

# $K3$ carpets

- $Y$ smooth regular surface, i.e. $H^1(\mathscr{O}_Y) = 0$.

**Definition**

A $K3$ carpet $\widetilde{Y}$ is a carpet on $Y$ such that $\omega_{\widetilde{Y}} \simeq \mathscr{O}_{\widetilde{Y}}$.

- $\widetilde{Y}$ proper over $\mathbf{C} \Rightarrow$ there exists $\omega_{\widetilde{Y}}$, Grothendieck dualizing sheaf on $\widetilde{Y}$ with nice functorial properties. ([4])
- $\widetilde{Y}$ locally Gorenstein $\Rightarrow \omega_{\widetilde{Y}}$ invertible. ([5])

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[4] Kleiman, *Relative duality for quasicoherent sheaves*, Compositio Math. **41** (1980), no. 1, 39–60.

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# Characterization of $K3$ carpets

- *$\widetilde{Y}$ is a K3 carpet* iff *$\mathscr{E} \simeq \omega_Y$.*

- *Let $\widetilde{Y}$ be a K3 carpet. Then $H^1(\mathscr{O}_{\widetilde{Y}}) = 0$.*

- $\{$ Non–split $K3$ carpets $\}/\sim \;\longleftrightarrow\; \mathbf{P}(\mathrm{Ext}^1_Y(\Omega_Y, \omega_Y))$.
- If $Y$ smooth Enriques surface, i.e. regular and $\omega_Y^2 = \mathscr{O}_Y$, then:
  - ⋆ $\dim \mathrm{Ext}^1_Y(\Omega_Y, \omega_Y) = h^{1,1} = 10$.

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- $X \xrightarrow{\pi} Y$, étale double cover.
  - $X$ smooth $K3$ surface: $\omega_X = \mathscr{O}_X,\ H^1(\mathscr{O}_X) = 0$.
  - $\pi_* \mathscr{O}_X = \mathscr{O}_Y \oplus \omega_Y$.
  - $\omega_Y$ trace zero module of $\pi$ $=$ conormal bundle of $K3$ carpets.

### “Moral” consequence:

- $K3$ carpets on Enriques surfaces, must be smoothable (like $K3$ carpets on rational normal scrolls ([6])).

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## Antecedents

- Double covers and double structures

  ⋆ Lung-Ying Fong, *Rational ribbons and deformation of hyperelliptic curves*, J. Algebraic Geom. **2** (1993), no. 2, 295–307.

  ⋆ Ribbons on $\mathbf{P}^1$ and hyperelliptic canonical morphisms.

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  ⋆ $K3$ carpets on rational normal scrolls and hyperelliptic smooth $K3$ surfaces.

- Finite covers and ropes on curves

  ⋆ M. González, *Smoothing of ribbons over curves*, J. reine angew. Math. **591** (2006), 201–235.

  ⋆ General infinitesimal theory relating first–order infinitesimal deformations of finite morphisms to ropes.

  ⋆ F.J. Gallego, M. González, and B.P. Purnaprajna, *Deformation of finite morphisms and smoothing of ropes*, arXiv:math.AG/0502467.

  ⋆ Smoothing of ropes of any multiplicity on curves with applications to ropes of multiplicity 3 on $\mathbf{P}^1$

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## Definition

- A smoothing for $\widetilde{Y}$ is a flat proper integral family $\mathscr{Y}$ over a smooth affine curve $T$, s. t.
    - for $0 \in T$, $\mathscr{Y}_0 = \widetilde{Y}$ and,
    - for $0 \neq t \in T$, $\mathscr{Y}_t$ is a smooth, irreducible, and, in our case, projective $K3$ surface.
- Embedded smoothing: Same as above, but
    - Ambient variety $Z$, $Y \subset \widetilde{Y} \subset Z$ (closed subschemes) and,
    - $\mathscr{Y} \subset Z \times T$ (closed subscheme).

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# Target result

**Theorem**

*Any projective K3 carpet* $\widetilde{Y}$ *on an Enriques* $Y$ *is smoothable.*

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*Let $\widetilde{Y}$ corresponding to $\tau \in \mathrm{Ext}^1(\Omega_Y, \omega_Y)$.*

- *$\widetilde{Y}$ is projective iff there exists an ample divisor $D$ on $Y$ s.t. $\int_D \tau = 0$, where $\tau \in H^{1,1}(Y) = H^2(Y, \mathbf{C})$.*
- *Projective K3 carpets on $Y$ are parametrized by countably infinitely many hyperplanes of the $9$-dimensional space $\mathbf{P}(\mathrm{Ext}^1(\Omega_Y, \omega_Y))$, corresponding to the classes in $NS(Y)$ of primitive ample divisors on $Y$.*

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*Let $Y \hookrightarrow \mathbf{P}^N$ with sectional genus $g$.*

- *Embedded K3 carpets in $\mathbf{P}^N$ supported on $Y$ are parametrized by a non–empty open set in*

$$\mathbf{P}(H^0(\mathcal{N}_{Y,\mathbf{P}^N} \otimes \omega_Y)),$$

*whose dimension is $g(N+1)+8$.*

## Strategy

- $Y \subset \widetilde{Y} \subset \mathbf{P}^N$, $\widetilde{Y}$ a $K3$ carpet.
- $X \xrightarrow{\pi} Y$, étale double covering s.t. $\pi_* \mathscr{O}_X = \mathscr{O}_Y \oplus \omega_Y$,
- $X$ smooth $K3$ surface.

$$\begin{tikzcd}
 & X \arrow[dl, "\pi"'] \arrow[dd, "\varphi"] & \\
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 & \mathbf{P}^N \arrow[rrrr, hook] \arrow[d] & & & & \mathbf{P}^N_T \arrow[d] \\
 & 0 \arrow[rrrr, hook] & & & & T
\end{tikzcd}$$

- $\mathscr{X}_0 = X$, $\Phi_0 = \varphi$, $\Phi_t$ embedding of a smooth $K3$ surface $\mathscr{X}_t$, $t \neq 0$.
- $\operatorname{im}(\Phi_t) = (\operatorname{im}\Phi)_t$, $t \neq 0$ smooth $K3$ surface.
- $Y = \operatorname{im}(\Phi_0) \subsetneqq (\operatorname{im}\Phi)_0$.
- $(\operatorname{im}\Phi)_0$ has same Hilbert polynomial as an embedded $K3$ carpet on $Y$. (reason: trace zero module of $\pi$ = conormal bundle of $K3$ carpets)
- So, if $\widetilde{Y} \subset (\operatorname{im}\Phi)_0$ then $\widetilde{Y} = (\operatorname{im}\Phi)_0$.

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 & X \arrow[rrrr, hook] \arrow[dl, "\pi"'] \arrow[dd, "\varphi"] & & & & \mathscr{X} \arrow[dl] \arrow[dd, "\Phi"] \\
Y \arrow[rr, hook] \arrow[dr, hook] & & \widetilde{Y} \arrow[r, equal, "?"] \arrow[dl, hook] & (\mathrm{im}\,\Phi)_0 \arrow[r, hook] \arrow[dll, hook] & \mathrm{im}\,\Phi \arrow[dr, hook] & \\
 & \mathbf{P}^N \arrow[rrrr, hook] \arrow[d] & & & & \mathbf{P}^N_T \arrow[d] \\
 & 0 \arrow[rrrr, hook] & & & & T
\end{tikzcd}$$

- $\mathscr{X}_0 = X$, $\Phi_0 = \varphi$, $\Phi_t$ embedding of a smooth $K3$ surface $\mathscr{X}_t$, $t \neq 0$.
- $\mathrm{im}\,(\Phi_t) = (\mathrm{im}\,\Phi)_t$, $t \neq 0$ smooth $K3$ surface.
- $Y = \mathrm{im}\,(\Phi_0) \subsetneqq (\mathrm{im}\,\Phi)_0$.
- $(\mathrm{im}\,\Phi)_0$ has same Hilbert polynomial as an embedded $K3$ carpet on $Y$. (reason: trace zero module of $\pi$ $=$ conormal bundle of $K3$ carpets)
- So, if $\widetilde{Y} \subset (\mathrm{im}\,\Phi)_0$ then $\widetilde{Y} = (\mathrm{im}\,\Phi)_0$.

# Strategy

$$\begin{tikzcd}
& X \arrow[dl, "\pi"'] \arrow[dd, "\varphi"] \arrow[rrrr, hook] & & & & \mathscr{X} \arrow[dl] \arrow[dd, "\Phi"] \\
Y \arrow[rr, hook] \arrow[dr, hook] & & \widetilde{Y} \arrow[r, equal, "?"] \arrow[dl, hook] & (\operatorname{im}\Phi)_0 \arrow[r, hook] \arrow[dll, hook] & \operatorname{im}\Phi \arrow[dr, hook] & \\
& \mathbf{P}^N \arrow[rrrr, hook] \arrow[d] & & & & \mathbf{P}^N_T \arrow[d] \\
& 0 \arrow[rrrr, hook] & & & & T
\end{tikzcd}$$

- $\mathscr{X}_0 = X, \quad \Phi_0 = \varphi, \quad \Phi_t$ embedding of a smooth $K3$ surface $\mathscr{X}_t, \; t \neq 0$.
- $\operatorname{im}(\Phi_t) = (\operatorname{im}\Phi)_t, \; t \neq 0$ smooth $K3$ surface.
- $Y = \operatorname{im}(\Phi_0) \subsetneqq (\operatorname{im}\Phi)_0$.
- $(\operatorname{im}\Phi)_0$ has same Hilbert polynomial as an embedded $K3$ carpet on $Y$. (reason: trace zero module of $\pi$ $=$ conormal bundle of $K3$ carpets)
- So, if $\widetilde{Y} \subset (\operatorname{im}\Phi)_0$ then $\widetilde{Y} = (\operatorname{im}\Phi)_0$.

# Embedded infinitesimal smoothing

- First–order infinitesimal deformation of $X \xrightarrow{\varphi} \mathbf{P}^N$:

$$\widetilde{X} \xrightarrow{\widetilde{\varphi}} \mathbf{P}^N_\Delta, \qquad (\Delta = \operatorname{Spec} k[\epsilon]/\epsilon^2)$$

such that

$$(\operatorname{im} \widetilde{\varphi})_0 = \widetilde{Y}.$$

## Theorem

*Let* $\widetilde{Y} \subset \mathbf{P}^N$ *be a projective K3 carpet. There exists a first–order infinitesimal deformation*

$$\widetilde{X} \xrightarrow{\widetilde{\varphi}} \mathbf{P}^N \times \Delta$$

*of* $X \xrightarrow{\varphi} \mathbf{P}^N$ *such that*

$$(\operatorname{im} \widetilde{\varphi})_0 = \widetilde{Y}.$$

- Since $\pi$ is étale, this follows from the general infinitesimal theory in ([7]).

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[7] M. González, *Smoothing of ribbons over curves*, J. reine angew. Math. **591** (2006), 201–235.

# Outline of the infinitesimal theory

- Normal sheaf of $\varphi$ defined by

$$0 \to \mathscr{T}_X \to \varphi^* \mathscr{T}_{\mathbf{P}^N} \to \mathscr{N}_\varphi \to 0$$

## Nice exact sequence

$$0 \to \mathscr{N}_\pi \to \mathscr{N}_\varphi \to \pi^* \mathscr{N}_{Y,\mathbf{P}^N} \to 0$$

## Key diagram

$$\begin{tikzcd}
H^0(\mathscr{N}_\varphi) \arrow[r, "\delta_1"] \arrow[d, "\Phi_1 \oplus \Phi_2"'] & \mathrm{Ext}^1(\Omega_X, \mathscr{O}_X) \arrow[d, "\Psi_1 \oplus \Psi_2"] \\
\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_Y) \oplus \mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E}) \arrow[r, "\delta_3"] & \mathrm{Ext}^1(\Omega_Y, \mathscr{O}_Y) \oplus \mathrm{Ext}^1(\Omega_Y, \mathscr{E})
\end{tikzcd}$$

# Outline of the infinitesimal theory

- Normal sheaf of $\varphi$ defined by

$$0 \longrightarrow \mathscr{T}_X \longrightarrow \varphi^* \mathscr{T}_{\mathbf{P}^N} \longrightarrow \mathscr{N}_\varphi \longrightarrow 0$$

$$\mathscr{N}_\varphi \xrightarrow{\sim} \pi^* \mathscr{N}_{Y,\mathbf{P}^N} \qquad (\mathscr{N}_\pi = 0)$$

## Key diagram

$$\begin{tikzcd}
H^0(\mathscr{N}_\varphi) \arrow[r, "\delta_1"] \arrow[d, equal, "\Phi_1 \oplus \Phi_2"'] & \mathrm{Ext}^1(\Omega_X, \mathscr{O}_X) \arrow[d, "\Psi_1 \oplus \Psi_2"] \\
\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_Y) \oplus \mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E}) \arrow[r, "\delta_3"] & \mathrm{Ext}^1(\Omega_Y, \mathscr{O}_Y) \oplus \mathrm{Ext}^1(\Omega_Y, \mathscr{E})
\end{tikzcd}$$

# Outline of the general infinitesimal theory

- $X \xrightarrow{\pi} Y \hookrightarrow Z$ , $X \xrightarrow{\varphi} Z$ , $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$.

$$\begin{tikzcd}
H^0(\mathcal{N}_\varphi) \arrow[r, "\delta_1"] \arrow[d, "\Phi_1 \oplus \Phi_2"'] & \operatorname{Ext}^1(\Omega_X, \mathcal{O}_X) \arrow[d, "\Psi_1 \oplus \Psi_2"] \\
\operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \oplus \operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) \arrow[r, "\delta_3"] & \operatorname{Ext}^1(\Omega_Y, \mathcal{O}_Y) \oplus \operatorname{Ext}^1(\Omega_Y, \mathcal{E}).
\end{tikzcd}$$

- $H^0(\mathcal{N}_\varphi) \longleftrightarrow \{$ $1^{\text{st}}$–order $\infty$-mal deformations $(\widetilde{X}, \widetilde{\varphi})$ of $(X, \varphi)$ $\}$.
- $\operatorname{Ext}^1(\Omega_X, \mathcal{O}_X) \longleftrightarrow \{\widetilde{X}\}$.
- $\operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{E}) \longleftrightarrow \{$ pairs $(\widetilde{Y}, \widetilde{i})$, $\widetilde{Y}$ rope conormal $\mathcal{E}$ and $\widetilde{Y} \xrightarrow{\widetilde{i}} Z$ extension morphism of $Y \overset{i}{\hookrightarrow} Z$$\}$.
- $\operatorname{Ext}^1(\Omega_Y, \mathcal{E}) \longleftrightarrow \{\widetilde{Y}\}$.
- $\operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \longleftrightarrow \{$ $1^{\text{st}}$–order deformations $\bar{Y} \subset Z \times \Delta$ of $Y \subset Z$$\}$.
- $\operatorname{Ext}^1(\Omega_Y, \mathcal{O}_Y) \longleftrightarrow \{\bar{Y}\}$.

# Outline of the general infinitesimal theory

$$\begin{tikzcd}
H^0(\mathscr{N}_\varphi) \arrow[r, "\delta_1"] \arrow[d, "\Phi_1 \oplus \Phi_2"'] & \mathrm{Ext}^1(\Omega_X, \mathscr{O}_X) \arrow[d, "\Psi_1 \oplus \Psi_2"] \\
\mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_Y) \oplus \mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E}) \arrow[r, "\delta_3"] & \mathrm{Ext}^1(\Omega_Y, \mathscr{O}_Y) \oplus \mathrm{Ext}^1(\Omega_Y, \mathscr{E}).
\end{tikzcd}$$

*Let* $(\widetilde{X}, \widetilde{\varphi}) = \nu \in H^0(\mathscr{N}_\varphi)$ *and* $(\widetilde{Y}, \widetilde{i}) = \Phi_2 \nu \in \mathrm{Hom}(\mathscr{I}/\mathscr{I}^2, \mathscr{E})$.

- *Then:*(8) $(\mathrm{im}\,\widetilde{\varphi})_0 = \mathrm{im}\,(\widetilde{Y} \xrightarrow{\widetilde{i}} Z)$.

---

[8] M. González, *Smoothing...*, Crelle **591** (2006)

## Smoothing of any first–order infinitesimal deformation of $\varphi$

**Theorem**

*Let* $Y \hookrightarrow \mathbf{P}^N$ *with sectional genus* $g$ *and* $N \leq 2g-1$*. Let* $\varphi : X \xrightarrow{\pi} Y \hookrightarrow \mathbf{P}^N$*. For any first–order infinitesimal deformation* $\widetilde{X} \xrightarrow{\widetilde{\varphi}} \mathbf{P}^N_\Delta$ *of* $\varphi$*, there exists a smooth proper family* $\mathscr{X} \to T$ *and* $\mathscr{X} \xrightarrow{\Phi} \mathbf{P}^N_T$*, over a smooth affine curve* $(T, 0)$*, s.t.*

- *the general fiber* $\mathscr{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N$*,* $t \in T - 0$*, is a closed immersion of a smooth K3 surface; and*
- *the fiber* $\mathscr{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}^N_\Delta$ *over the tangent vector at* $0 \in T$ *is* $\widetilde{X} \xrightarrow{\widetilde{\varphi}} \mathbf{P}^N_\Delta$*.*

# Embedded smoothing

## Theorem

*Let* $\widetilde{Y} \subset \mathbf{P}^N$ *projective K3 carpet, on* $Y \subset \mathbf{P}^N$ *with* $N \leq 2g-1$*.*

*There exists* $\mathscr{X} \xrightarrow{\Phi} \mathbf{P}^N_T$ *as above, s.t. for* $\mathscr{Y} := \operatorname{im}\Phi$

- *the general fiber* $\mathscr{Y}_t$, $t \in T - 0$, *is a smooth non–degenerate K3 surface in* $\mathbf{P}^N$,
- *the central fiber* $\mathscr{Y}_0 \subset \mathbf{P}^N$ *is* $\widetilde{Y} \subset \mathbf{P}^N$.

Skip

- Let $Y \subset \widetilde{Y} \subset \mathbf{P}^N$ and $\widetilde{X} \xrightarrow{\widetilde{\varphi}} \mathbf{P}^N_\Delta$ s.t. $\widetilde{Y} = (\operatorname{im} \widetilde{\varphi})_0$.
- Let $\mathscr{X} \xrightarrow{\Phi} \mathbf{P}^N_T$, s.t.
  - $\mathscr{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N, t \neq 0$ embedding of a smooth $K3$ surface,
  - $(\mathscr{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}^N_\Delta) = (\widetilde{X} \xrightarrow{\widetilde{\varphi}} \mathbf{P}^N_\Delta)$, $(\mathscr{X}_0 \xrightarrow{\Phi_0} \mathbf{P}^N) = (X \xrightarrow{\varphi} \mathbf{P}^N)$.

$$\begin{tikzcd}
 & X \arrow[rrr, hook] \arrow[dl, "\pi"'] \arrow[dd, "\varphi"] & & & \widetilde{X} \arrow[rr, hook] \arrow[dl] \arrow[dd, "\widetilde{\varphi}"] & & \mathscr{X} \arrow[dl] \arrow[dd, "\Phi"] \\
Y \arrow[rr, hook] \arrow[dr, hook] & & \widetilde{Y} = (\operatorname{im} \widetilde{\varphi})_0 \arrow[r, hook] \arrow[dl, hook] \arrow[dl, hook, bend left] & \operatorname{im} \widetilde{\varphi} \arrow[rr, hook] \arrow[dr, hook] & & \mathscr{Y} \arrow[dr, hook] & \\
 & \mathbf{P}^N \arrow[rrr, hook] \arrow[d] & & & \mathbf{P}^N_\Delta \arrow[rr, hook] \arrow[d] & & \mathbf{P}^N_T \arrow[d] \\
 & 0 \arrow[rrr, hook] & & & \Delta \arrow[rr, hook] & & T
\end{tikzcd}$$

- Let $Y \subset \widetilde{Y} \subset \mathbf{P}^N$ and $\widetilde{X} \xrightarrow{\widetilde{\varphi}} \mathbf{P}^N_\Delta$ s.t. $\widetilde{Y} = (\operatorname{im}\widetilde{\varphi})_0$.
- Let $\mathscr{X} \xrightarrow{\Phi} \mathbf{P}^N_T$, s.t.
  - $\mathscr{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N, t \neq 0$ embedding of a smooth $K3$ surface,
  - $(\mathscr{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}^N_\Delta) = (\widetilde{X} \xrightarrow{\widetilde{\varphi}} \mathbf{P}^N_\Delta)$, $(\mathscr{X}_0 \xrightarrow{\Phi_0} \mathbf{P}^N) = (X \xrightarrow{\varphi} \mathbf{P}^N)$.
- Then, $\mathscr{Y} := \operatorname{im}\Phi,$ is $T$–flat,
  - $\mathscr{Y}_t,\ t \neq 0$ is a smooth $K3$ surface, and

$$\begin{tikzcd}
 & X \arrow[rrr, hook] \arrow[dl, "\pi"'] \arrow[dd, "\varphi"] & & & \widetilde{X} \arrow[rr, hook] \arrow[dl] \arrow[dd, "\widetilde{\varphi}"] & & \mathscr{X} \arrow[dl] \arrow[dd, "\Phi"] \\
Y \arrow[rr, hook] \arrow[dr, hook] & & \widetilde{Y} = (\operatorname{im}\widetilde{\varphi})_0 \arrow[r, hook] \arrow[dl, hook] \arrow[dl, hook, bend left] & \operatorname{im}\widetilde{\varphi} \arrow[rr, hook] \arrow[dr, hook] & & \mathscr{Y} \arrow[dr, hook] & \\
 & \mathbf{P}^N \arrow[rrr, hook] \arrow[d] & & & \mathbf{P}^N_\Delta \arrow[rr, hook] \arrow[d] & & \mathbf{P}^N_T \arrow[d] \\
 & 0 \arrow[rrr, hook] & & & \Delta \arrow[rr, hook] & & T
\end{tikzcd}$$

- Let $\mathscr{X} \xrightarrow{\Phi} \mathbf{P}^N_T$, s.t.
  - $\mathscr{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N, t \neq 0$ embedding of a smooth $K3$ surface,
  - $(\mathscr{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}^N_\Delta) = (\widetilde{X} \xrightarrow{\widetilde{\varphi}} \mathbf{P}^N_\Delta)$, $(\mathscr{X}_0 \xrightarrow{\Phi_0} \mathbf{P}^N) = (X \xrightarrow{\varphi} \mathbf{P}^N)$.
- Then, $\mathscr{Y} := \operatorname{im}\Phi,$ is $T$–flat,
  - $\mathscr{Y}_t,\ t \neq 0$ is a smooth $K3$ surface, and
  - $\widetilde{Y} = (\operatorname{im}\widetilde{\varphi})_0 \subset \mathscr{Y}_0$.

$$\begin{tikzcd}
 & X \arrow[rrrr, hook] \arrow[dl, "\pi"'] \arrow[dd, dashed, "\varphi"] & & & & \widetilde{X} \arrow[rr, hook] \arrow[dl] \arrow[dd, dashed, "\widetilde{\varphi}"] & & \mathscr{X} \arrow[dl] \arrow[dd, "\Phi"] \\
Y \arrow[rr, hook] \arrow[dr, hook] & & \widetilde{Y} = (\operatorname{im}\widetilde{\varphi})_0 \arrow[rr, hook] \arrow[dl, hook] \arrow[dl, hook, bend left] & & \operatorname{im}\widetilde{\varphi} \arrow[rr, hook] \arrow[dr, hook] & & \mathscr{Y} \arrow[dr, hook] & \\
 & \mathbf{P}^N \arrow[rrrr, hook] \arrow[d] & & & & \mathbf{P}^N_\Delta \arrow[rr, hook] \arrow[d] & & \mathbf{P}^N_T \arrow[d] \\
 & 0 \arrow[rrrr, hook] & & & & \Delta \arrow[rr, hook] & & T
\end{tikzcd}$$

- Let $\mathscr{X} \xrightarrow{\Phi} \mathbf{P}^N_T$, s.t.
  - $\mathscr{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N, t \neq 0$ embedding of a smooth $K3$ surface,
  - $(\mathscr{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}^N_\Delta) = (\widetilde{X} \xrightarrow{\widetilde{\varphi}} \mathbf{P}^N_\Delta)$, $(\mathscr{X}_0 \xrightarrow{\Phi_0} \mathbf{P}^N) = (X \xrightarrow{\varphi} \mathbf{P}^N)$.
- Then, $\mathscr{Y} := \operatorname{im}\Phi,$ is $T$–flat,
  - $\mathscr{Y}_t,\ t \neq 0$ is a smooth $K3$ surface, and
  - $\widetilde{Y} = (\operatorname{im}\widetilde{\varphi})_0 \subset \mathscr{Y}_0$, and
  - trace zero module of $\pi =$ conormal bundle of $K3$ carpets. Then

$$\begin{tikzcd}
 & X \arrow[rrr, hook] \arrow[dl, "\pi"'] \arrow[dd, "\varphi"] & & & \widetilde{X} \arrow[rr, hook] \arrow[dl] \arrow[dd, "\widetilde{\varphi}"] & & \mathscr{X} \arrow[dl] \arrow[dd, "\Phi"] \\
Y \arrow[rr, hook] \arrow[dr, hook] & & \widetilde{Y} = (\operatorname{im}\widetilde{\varphi})_0 \arrow[r, hook] \arrow[dl, hook] \arrow[dl, hook, bend left] & \operatorname{im}\widetilde{\varphi} \arrow[rr, hook] \arrow[dr, hook] & & \mathscr{Y} \arrow[dr, hook] & \\
 & \mathbf{P}^N \arrow[rrr, hook] \arrow[d] & & & \mathbf{P}^N_\Delta \arrow[rr, hook] \arrow[d] & & \mathbf{P}^N_T \arrow[d] \\
 & 0 \arrow[rrr, hook] & & & \Delta \arrow[rr, hook] & & T
\end{tikzcd}$$

- Let $\mathscr{X} \xrightarrow{\Phi} \mathbf{P}^N_T$, s.t.
  - $\mathscr{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N, t \neq 0$ embedding of a smooth $K3$ surface,
  - $(\mathscr{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}^N_\Delta) = (\widetilde{X} \xrightarrow{\widetilde{\varphi}} \mathbf{P}^N_\Delta)$, $(\mathscr{X}_0 \xrightarrow{\Phi_0} \mathbf{P}^N) = (X \xrightarrow{\varphi} \mathbf{P}^N)$.
- Then, $\mathscr{Y} := \mathrm{im}\,\Phi$, is $T$–flat,
  - $\mathscr{Y}_t,\ t \neq 0$ is a smooth $K3$ surface, and
  - $\widetilde{Y} = (\mathrm{im}\,\widetilde{\varphi})_0 \subset \mathscr{Y}_0$, and
  - $(\mathrm{im}\,\widetilde{\varphi})_0$ and $\mathscr{Y}_0$ have same Hilbert polynomial. So

$$\begin{tikzcd}
 & X \arrow[rrrr, hook] \arrow[dl, "\pi"'] \arrow[dd, "\varphi"] & & & & \widetilde{X} \arrow[rr, hook] \arrow[dl] \arrow[dd, "\widetilde{\varphi}"] & & \mathscr{X} \arrow[dl] \arrow[dd, "\Phi"] \\
Y \arrow[rr, hook] \arrow[dr, hook] & & \widetilde{Y} \arrow[r, phantom, "="] \arrow[dl, hook] & (\mathrm{im}\,\widetilde{\varphi})_0 \arrow[r, hook] \arrow[dll, hook] & \mathrm{im}\,\widetilde{\varphi} \arrow[rr, hook] \arrow[dr, hook] & & \mathscr{Y} \arrow[dr, hook] & \\
 & \mathbf{P}^N \arrow[rrrr, hook] \arrow[d] & & & & \mathbf{P}^N_\Delta \arrow[rr, hook] \arrow[d] & & \mathbf{P}^N_T \arrow[d] \\
 & 0 \arrow[rrrr, hook] & & & & \Delta \arrow[rr, hook] & & T
\end{tikzcd}$$

- Let $\mathscr{X} \xrightarrow{\Phi} \mathbf{P}^N_T$, s.t.
  - $\mathscr{X}_t \xrightarrow{\Phi_t} \mathbf{P}^N, t \neq 0$ embedding of a smooth $K3$ surface,
  - $(\mathscr{X}_\epsilon \xrightarrow{\Phi_\epsilon} \mathbf{P}^N_\Delta) = (\widetilde{X} \xrightarrow{\widetilde{\varphi}} \mathbf{P}^N_\Delta)$, $(\mathscr{X}_0 \xrightarrow{\Phi_0} \mathbf{P}^N) = (X \xrightarrow{\varphi} \mathbf{P}^N)$.
- Then, $\mathscr{Y} := \operatorname{im}\Phi$, is $T$–flat,
  - $\mathscr{Y}_t,\ t \neq 0$ is a smooth $K3$ surface, and
  - $\widetilde{Y} = (\operatorname{im}\widetilde{\varphi})_0 = \mathscr{Y}_0$.

$$\begin{tikzcd}
 & X \arrow[rrr, hook] \arrow[dl, "\pi"'] \arrow[dd, "\varphi"] & & & \widetilde{X} \arrow[rr, hook] \arrow[dl] \arrow[dd, "\widetilde{\varphi}"] & & \mathscr{X} \arrow[dl] \arrow[dd, "\Phi"] \\
Y \arrow[rr, hook] \arrow[dr, hook] & & \widetilde{Y} = (\operatorname{im}\widetilde{\varphi})_0 \arrow[r, hook] \arrow[dl, hook] \arrow[dl, hook, bend left] & \operatorname{im}\widetilde{\varphi} \arrow[rr, hook] \arrow[dr, hook] & & \mathscr{Y} \arrow[dr, hook] & \\
 & \mathbf{P}^N \arrow[rrr, hook] \arrow[d] & & & \mathbf{P}^N_\Delta \arrow[rr, hook] \arrow[d] & & \mathbf{P}^N_T \arrow[d] \\
 & 0 \arrow[rrr, hook] & & & \Delta \arrow[rr, hook] & & T
\end{tikzcd}$$

## The Hilbert point of an embedded $K3$ carpet

**Theorem**

*Let $\widetilde{Y} \subset \mathbf{P}^N$ be a projective $K3$ carpet embedded as above $(N \leq 2g - 1)$. Then the Hilbert point of $\widetilde{Y}$ is nonsingular.*