Algebraic Cycles and Singularities of Normal Functions

joint work in progress with Mark Green

Talk given at Universidad de Villadolid - Aug 16, 2006

Outline

I. Hodge conjecture (HC)

Classical approach; summary of to

II Classical normal functions

III Extended normal functions

$$\begin{cases} \text{Hodge class } \zeta \longmapsto \nu_\zeta \\ \text{HC} \Leftrightarrow \operatorname{sing} \nu_\zeta \neq \phi \end{cases}$$

IV Hodge-theoretic classifying maps $\rho : S \to \overline{\mathcal{M}}_H$

$$\begin{cases} \operatorname{sing} \nu_\zeta = \text{component of } \rho^{-1}(\mathcal{B}) \\ \rho^*([\mathcal{B}]) \neq 0 \Rightarrow \text{existence thm} \end{cases}$$

V Dimension counts and excess intersection formulas

I. Hodge conjecture (HC)

- (X, L) is a smooth variety $/\mathbb{C}$ and $L \to X$ is a very ample line bundle

- $$H^r(X, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X)$$
  $$\overline{H^{p,q}(X)} = H^{q,p}(X)$$

$\rightsquigarrow$ $$H_g^r(X) = H^{r,r}(X) \cap H^{2r}(X, \mathbb{Z})$$

- $Z^r(X) =$ codimension r algebraic cycles $Z = \sum_i n_i Z_i$

- Fundamental class

$$(*) \qquad \begin{array}{ccc} Z^r(X) & \longrightarrow & Hg^r(X) \\ \Psi & & \Psi \\ Z & \longrightarrow & [Z] \end{array}$$

<u>HC (original form):</u> <u>(*) is onto</u>

(i) True for $r=1$ (Lefschetz)

(ii) False for torsion and $r \geq 2$

- Atiyah-Hirzebruch
- Soulé-Voisin-Kollar

(iii) Two proofs for $r=1$

- Poincaré-Lefschetz
- Kähler (Kodaira-Spencer)

2nd is false for $r \geq 2$, Voisin

Classical approach; Summary

- induction on dim X
- reduce to dim $X = 2n$ and

$$Hg^n(X)_{prim}$$

- fibre X by $X_s \in |L|$

$$\{X_s\}_{s \in |L|} = \mathcal{X} \subset X \times |L|$$
$$\downarrow$$
$$|L|$$

$\rightsquigarrow$ classical $\left\{\begin{array}{l} \text{- normal functions } \nu_\zeta \\ \text{- Lefschetz proof when } n=1 \\ \quad \text{(restrict to Lefschetz pencils)} \end{array}\right.$

more recent $\left\{\begin{array}{l} \text{- extended normal functions} \\ \quad \text{and their singularities} \end{array}\right.$

- HC $\Leftrightarrow$ sing $\nu_\zeta \neq \phi$ for $L \gg 0$

- sing $\nu_\zeta \subset D_{\text{sing}} \subset |L|$

  $\Rightarrow$ codim (sing ν_ζ) $\overset{?}{=} 2$

more recent

$\rightsquigarrow$ $\boxed{\text{codim(sing } \nu_\zeta) = \text{? for } L \gg 0}$

- sing ν_ζ = component of $\rho^{-1}(\mathcal{B})$

$n=1$ codim (sing $\underline{\nu}_\zeta$) = codim $\mathcal{B}$

(here (X, ζ) varies in $\mathcal{M}_\zeta \subset \mathcal{M}$)

$\Rightarrow \rho^*([\mathcal{B}]) \neq 0$

Assuming HC

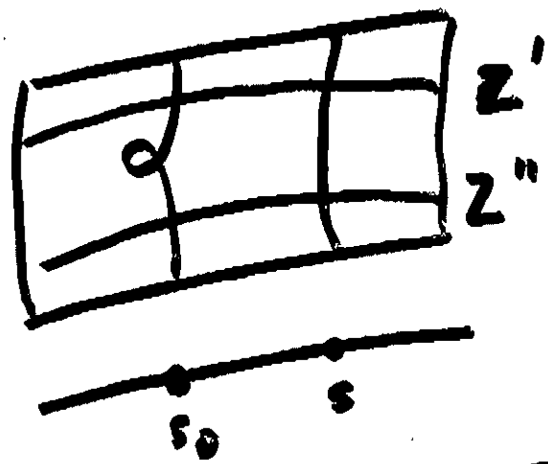
$n=2$ $\boxed{\begin{array}{c}\text{codim(sing } \underline{\nu}_\zeta) = \text{"codim"} \mathcal{B} + \delta \\ \delta > 0\end{array}}$

Here $\delta \sim c \dim |L|$ and "codim" is an I-transversality dimension count

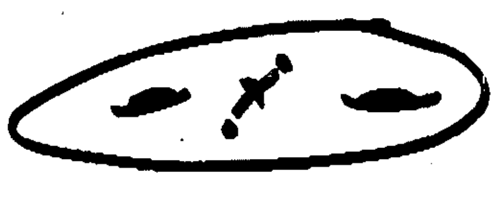
II. Classical normal functions

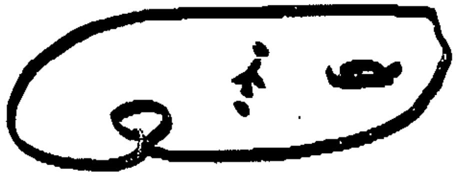
Motivation: $|X_s|$ is Lefschetz pencil on a surface, Z is an algebraic 1-cycle with

$$\begin{cases} Z_s = Z \cdot X_s \\ \deg Z_s = 0 \end{cases}$$



$Z = Z' - Z''$

$X_s =$ 

$X_{s_0} =$ 

$$w \mapsto \nu_Z(s) = AJ_{X_s}(Z_s) \in J(X_s)$$

$$\| \\ \mathrm{Pic}^0(X_s)$$

Works also for s_0 where

$$1 \to \mathbb{C}^* \to J_e(X_{s_0}) \to J(\hat{X}_{s_0}) \to 0$$

$$\| \\ \mathrm{Pic}^0(X_{s_0})$$

- ν_Z depends only on

$$\zeta = [Z] \in Hg^1(X)_{prim}$$

- Given any $\zeta \in Hg^1(X)_{prim}$ we may construct ν_ζ

- Jacobi inversion with dependence on parameters gives

$$\nu_\zeta(s) = AJ_{X_s}(Z_s)$$

where Z_s, $s \in \mathbb{P}^1$, traces out $Z \in Z^2(X)$ with $[Z] = \zeta$

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Assume given VHS of weight $2n-1$ $(\mathcal{H}_{\mathbb{Z}}, \mathcal{F}^\bullet, \nabla)$ over $S^* = S \setminus D$ where D has local normal crossings and unipotent monodromies

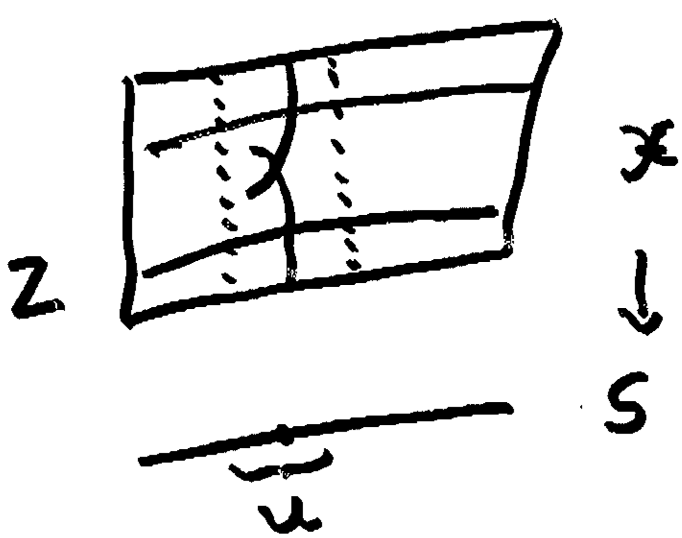
$\rightsquigarrow$ canonical extension

$$\begin{cases} (\mathcal{H}_{\mathbb{Z},e}, \mathcal{F}_e^p, \nabla) \\ \mathcal{F}_e^p \xrightarrow{\nabla} \mathcal{F}_e^{p-1} \otimes \Omega_S^1(\log D) \end{cases}$$

$\rightsquigarrow$ $0 \to \mathcal{H}_{\mathbb{Z},e} \to \check{\mathcal{F}}_e^n \longrightarrow \mathcal{J}_e \to 0$

Defn: A (classical) <u>normal function</u> ν is a section of $\mathcal{J}_{e,\nabla}$

(means $\nabla \tilde{\nu} \in \check{\mathcal{F}}_e^{n+1} \otimes \Omega_S^1((\log D))$)



$\begin{cases} [Z_u] = 0 \text{ in} \\ H^{2n}(\mathcal{X}_u, \mathbb{Z}) \end{cases}$

$$\Rightarrow Z_s = \partial \Gamma_s \;,\; (T_i - I)\Gamma_s = 0$$

$$\nu_Z(s)(\omega(s)) = \int_{\Gamma_s} \omega(s), \quad \omega \in \mathcal{F}^n_{e,s}$$

<u>Remark</u>: $\dim S = 1$ and $[Z_s] = 0$ in $H^{2n}(X_s, \mathbb{Z})$, $s \in S^*$, then Clemens-Schmid implies $[mZ_u] = 0$ as above

<u>Ex</u>: $\zeta \in Hg^n(\mathcal{X})$ and $\zeta_u = 0$ in $H^{2n}(X_u, \mathbb{Z})$

$$\rightsquigarrow \nu_\zeta \in \Gamma(S, \mathcal{J}_{e,\nabla})$$

(El Zein-Zucker). If $\dim S = 1$ and $\zeta_s = 0$ as above then

$$m\zeta_u = 0$$

<u>Conclusion: Modulo torsion, primitive Hodge classes give normal functions over 1-dimensional bases</u>

Due to the failure of Jacobi inversion for $n \geqq 2$ this result has been of limited use. It was realized in the 1970's that to use normal functions one needed to

(i) go to higher dimensional bases

(ii) work modulo torsion (related to (i))

(iii) deal with singularities of ν_ζ - in fact, <u>these are of central interest</u>

Since then have

- General VHS (Cattani-Kaplan-Schmid, Kashiwara, M. Saito)
- Decomposition theorem (B-B-D-G + de Cataldo-Migliorini)
- Partial compactifications of $\Gamma\backslash D$'s (Kato-Usui)

$\rightsquigarrow \longrightarrow \cdots$

III. Extended normal functions

(ENF's)

Defn: Given by a section ν of $\mathcal{J}_\nabla$ over S^* satisfying

(i) growth condition along $D = \cup D_i$

(ii) $(T_i - I)\, m\tilde{\nu} = (T_i - I)\,\delta_i$

around $s_0 \in D$, where

$\tilde{\nu}$ = lift of ν in $U^* = U \cap S^*$,

$\delta_i \in H^{2n-2}(X_{\underline{s}}, \mathbb{Z})$ where $\underline{s} \in U^*$

$$\rightsquigarrow \boxed{0 \to \mathcal{J}_{e,\nabla} \to \tilde{\mathcal{J}}_{e,\nabla} \xrightarrow{\sigma} \mathcal{G} \to 0}$$

Defn: $\operatorname{sing} \nu = \operatorname{support}(\sigma(\nu)) \in \mathcal{G}_{\mathbb{Q}}$

<u>Remark:</u> (Pearlstein et al.)

$$\text{ENF} \longleftrightarrow \left\{ \begin{array}{l} \text{2 step admissable} \\ \text{VMHS} \quad (\text{M. Saito}) \end{array} \right\}$$

<u>Theorem:</u> For $j : S^{\vee} \longrightarrow S$

$$(\mathcal{G}_{\mathbb{Q}})_{s_0} \cong \text{"Tate part" of } (IH^1(j_* \mathcal{H}_{\mathbb{Q}}))$$

$$\cong \text{"Tate part" of } H^2(B^{\bullet}_{s_0})$$

where $B^{\bullet}_{s_0}$ is complex constructed from $N_i = \log T_i$

<u>Theorem:</u> For (X, L), $S = \widetilde{|L|}$

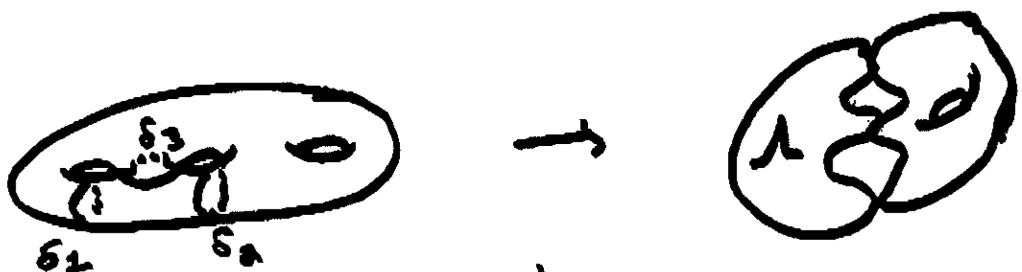
$$Hg^n(X)_{\text{prim}} \longleftrightarrow \text{ENF's} / J(X)$$

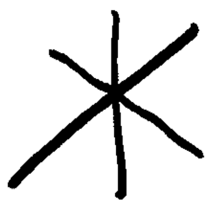
<u>Corollary:</u> HC $\Longleftrightarrow$ $\operatorname{sing} \nu_{\zeta} \neq \phi$ for $L \gg 0$

<u>Ex</u>: $\Lambda \subset X \subset \mathbb{P}^3$, $\deg X = 4$

$$Z = H - 4\Lambda$$

$$\rightsquigarrow \operatorname{sing}^\vee Z = \Lambda^\perp \subset \check{\mathbb{P}}^3$$



$\mathbb{P}^2 \cap D =$ 

(quasi-local normal crossings)

<u>Note</u>: Have vanishing cycles $\delta_1, \delta_2, \delta_3$ with

$$\delta_1 + \delta_2 + \delta_3 \sim 0$$

HC $\Rightarrow$ this is general phenomenon

IV Hodge-theoretic classifying maps

Given a lattice $H_{\mathbb{Z}} \cong \mathbb{Z}^{2h}$ with unimodular symplectic form Q and $\underline{h} = (h^{2n-1,0}, \ldots, h^{n,n-1})$ with

$$h = h^{2n-1,0} + \cdots + h^{n,n-1}$$

we denote by

$$D_{\underline{h}} \cong G/H \quad , \quad G = \mathrm{Aut}(H_{\mathbb{R}}, Q$$

the space of polarized Hodge structures $\{F^p\}$ on $H_{\mathbb{C}}$ with the given Hodge numbers, and by

$$\mathcal{A}_{\underline{h}} = \Gamma \backslash D_{\underline{h}} \quad , \quad \Gamma = \mathrm{Aut}(H_{\mathbb{Z}}, Q)$$

the moduli space of (equivalence classes of) polarized Hodge structures of weight $2m-1$. We also denote by

$$J_{\underline{h}} \to \mathcal{A}_{\underline{h}}$$

the universal family (actually, a stack) of polarized intermediate Jacobians

Kato - Usui (work in progress) have defined partial compactification

$$\mathcal{A}_{\underline{h},\Sigma} = \Gamma \backslash D_{\underline{h},\Sigma}$$

where Σ is a <u>fan</u> given by a collection of rational nilpotent cones

$$\sigma = \sigma(\underline{N}) = \sigma(N_1, .., N_\ell) \subset \mathfrak{g}_{\mathbb{Q}}$$

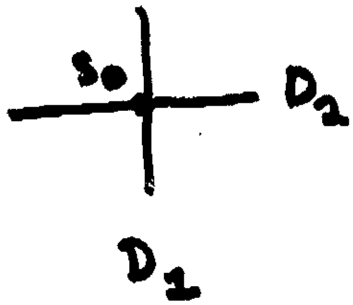
where

$$\begin{cases} N_i^{2m} = 0 \\ [N_i, N_j] = 0 \end{cases}$$

<u>Ex</u>: Semi-stable reduction applied to $\mathcal{X} \to |L|$ gives

$$\tilde{\mathcal{X}} \to S$$

where the discriminant locus $D \subset S$ has local normal crossings and each $s_0 \in D$



gives a $\sigma(\underline{N})$ where

$$\begin{cases} N_i = \log T_i \\ T_i = \text{monodromy around } D_i. \end{cases}$$

These $\sigma(\underline{N})$'s generate a fan and there is a regular Torelli map

$$S \xrightarrow{\tau} \mathcal{A}_{\underline{h},\Sigma}$$

Given a Hodge class ζ as above we have

$$\begin{tikzcd} & \mathcal{J}_{\underline{h},\Sigma} \arrow[d] \\ {|L|} \arrow[ur, dashed, "\nu_\zeta"] \arrow[r, dashed, "\tau"] & \mathcal{A}_{\underline{h},\Sigma} \end{tikzcd}$$

not yet defined in general

where the dotted arrows denote "meromorphic" maps

<u>Ex</u>: Assuming the HC and passing to a multiple of ζ, we may find a smooth

$$\begin{cases} W^n \subset X \\ [W]_{prim} = \zeta \end{cases}$$

then for $L \gg 0$ a general

$$X_{s_0} \in |\mathcal{L}_W(L)|$$

will have nodes $p_1, \dots, p_\ell$ corresponding to vanishing cycles

$$\delta_i \in H_{2n-1}(X_s, \mathbb{Z})$$

as $s \to s_0$. There is one relation

$$\delta_1 + \dots + \delta_\ell \sim 0$$

and we denote by

$$B(\sigma) \subset \mathcal{Q}_{h, \Sigma}$$

the corresponding boundary

component. We then have

<u>**Theorem**</u>: $s_0 \in \operatorname{sing} \nu_\xi$ <u>and</u>

$$(\operatorname{sing} \nu_\xi)_{s_0} = (\tau^{-1}(B(\sigma)))_{s_0}$$

In this example

$n=1$ $$\operatorname{codim}_{\mathcal{O}_{\underline{h},\Sigma}} (B(\sigma)) = \ell$$

$n=2$ $$\operatorname{codim}_{\mathcal{O}_{\underline{h},\Sigma}} (B(\sigma)) = \ell + \ell(h^{3,0}-1)$$

This leads to

V Dimension counts and excess intersection formulas

classical $n=1$ case For $L \gg 0$ we h

Theorem: $\operatorname{codim}_{|L|} \left(\pi^{-1}(B)\right)_{s_0} = \ell - h^{2,0}(X)$

Suggests let

$$\mathcal{M} = \left\{ \begin{array}{l} \text{moduli space for} \\ X \text{ - assumed to exist} \end{array} \right\}$$

$$\cup$$

$$\mathcal{M}_\zeta = \left\{ \begin{array}{l} \text{locus where } \zeta \text{ is a} \\ \text{Hodge class - algebrai} \\ \text{by D-C-C} \end{array} \right.$$

$\rightsquigarrow$ "expected" $\operatorname{codim}_{\mathcal{M}}(\mathcal{M}_\zeta) = h^{2,0}$

We assume this

For $\{\gamma_t\}_{t\in\mathcal{M}}$ we let

$$\begin{aligned} |\gamma| &= \{|L_t| : t\in\mathcal{M}\} \\ &\cup \\ |\gamma|_\zeta &= \{|L_t| : t\in\mathcal{M}_\zeta\} \end{aligned}$$

then

$$\begin{tikzcd} {|\gamma|} \arrow[r, dashed, "\Phi"] & \mathcal{A}_{\underline{h},\Sigma} \\ {|\gamma|_\zeta} \arrow[u, phantom, "\cup"] \arrow[ur, dashed, "\Phi_\zeta"'] & \end{tikzcd}$$

and

$$\Phi^{-1}(\mathcal{B}) = \Phi_\zeta^{-1}(\mathcal{B})_{s_0} \subseteq |\gamma|_\zeta$$

<u>**Theorem:** The map $|\gamma| \overset{\Phi}{\dashrightarrow} \mathcal{A}_{\underline{h},\Sigma}$ is defined and transverse relative to $\mathcal{B}$ near s_0 for</u> $L \gg 0$

Corollary: $\Phi_\zeta^*([B]) = [\tau_\zeta^{-1}(B)] \wedge c_{top}(\mathcal{H}^{0,2})$

RHS is "usually" non-zero

(*) $$\Rightarrow \Phi_\zeta^*([B]) \neq 0$$

An a priori proof of (*) would give another proof of the Lefschetz (1,1) theorem over a component of $\mathcal{M}_\zeta$.

Remarks. Result is meant to be illustrative, not definitive.

Need to extend to

$$\begin{tikzcd}
 & \mathcal{J}_{h,\Sigma} \arrow[d] \\
|\mathcal{L}|_\zeta \arrow[ur, dashed, "\Psi_\zeta"] \arrow[r, dashed, "\Phi_\zeta"] & \mathcal{A}_{h,\Sigma}
\end{tikzcd}$$

Before turning to the $n=2$ case need to mention

<u>I - transversality</u>. Given

$$\begin{array}{ccc} M & \xrightarrow{F} & A \\ \cup & & \cup \\ N & \rightarrow & B \end{array} \qquad F^{-1}(B) = N$$

$$\Rightarrow \text{codimension}_M (N) \leq \text{codim}_A (B)$$

Now suppose we have $I \subset TA$

$$\begin{cases} F_* : TM \rightarrow I \\ I \cap TB \text{ transverse} \end{cases}$$

$$\Rightarrow \text{codim}_M (N) \leq \text{codim}_I (I \cap TB)$$

Refinement due to integrability usually, but not always, needed.

<u>$n=2$ case</u> First issue is

$$\operatorname{codim}_{\mathcal{M}}(\mathcal{M}_\zeta) = ?$$

- $h^{4,0} + h^{3,1}$ not correct because $(\zeta')^{0,4} = 0$

- $h^{3,1}$ not correct because of integrability

For $T_\zeta = \{\theta \in H^1(\Theta_X) : \theta \cdot \zeta = 0\}$ set

$$\sigma_\zeta = \dim\{\operatorname{Im} H^{4,0}(X) \otimes T_\zeta \to H^{3,1}(X)\}$$

<u>**Theorem:**</u> $\operatorname{codim}_{\mathcal{M}}(\mathcal{M}_\zeta) \leqq h^{3,1} - \sigma_\zeta$.

<u>Equality holds in examples</u>

Example of I-transversality with integrability conditions - for $\underline{\hat{h}} = (h^{4,0}, h^{3,1}, h^{2,2})$ have

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{A}_{\underline{\hat{h}}} \\ \cup & & \cup \\ \mathcal{M}_Z & \rightarrow & \mathcal{A}_{\underline{\hat{h}},Z} \end{array} \quad \leftarrow \boxed{\text{here the codimension is } h^{4,0}+h^{3,1}}$$

$\rightsquigarrow$ $\mathcal{M}$ does not meet $\mathcal{A}_{\underline{\hat{h}},Z}$ transversely because of $T\mathcal{M} \to I \subset T\mathcal{A}_{\underline{\hat{h}}}$ - must also take into account $[I,I]$, $[I,[I,I]]$, ...

Now <u>assume</u> HC

$$\begin{array}{ccc} |L| & \overset{\tau}{\dashrightarrow} & \bar{\mathcal{A}}_{\underline{h},\Sigma} \\ \cup & & \cup \\ (\mathrm{sing}\,\nu_{\zeta}) & \longrightarrow & B \end{array}$$

<u>Theorem:</u> For $L \gg 0$ (relative to W)

$$\operatorname*{codim}_{|L|} \left(\tau^{-1}(B)\right)_{s_0} = \ell - (h^{3,1} - \sigma_{\zeta}) - \delta$$

<u>where</u>

$$\delta = \dim\Big(\ker\{H^2(\Lambda^2 N_{W/X}(L^{ds_0})) \to H^2(N_{u}$$

(thus $\delta \gg 0$ for $L \gg 0$)

<u>Note</u>: $\operatorname*{codim}_{\bar{\mathcal{A}}_{\underline{h},\Sigma}} (B) = \ell + h^{3,0}(\ell - 1)$

and last term drops out by

I-transversality - no [I,I] conditi

Letting (X, ζ) vary

$$\operatorname*{codim}_{|\mathcal{L}|}\left(\Phi^{-1}(B)\right) = \ell - \delta \quad , \delta > 0 .$$

<u>Corollary: Assuming HC, Φ can **never** be transverse for $L \gg 0$</u>

The EIF is

$$\boxed{\Phi_\zeta^*([B]) = Gy\left(c_{top}(\mathcal{E}') \wedge c_{top}(\mathcal{E}'') \wedge c_{top}(\mathcal{E}''')\right)}$$

where Gy is the Gysin map for (desingularized) $\Phi^{-1}(B)_{s_0} \hookrightarrow |\mathcal{L}|$

and

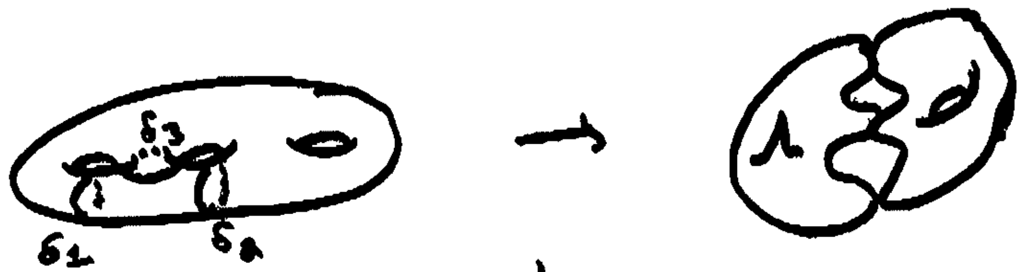
$$\begin{cases} \operatorname{rank} \mathcal{E}' = h^{3,1} - \sigma_3 \\ \operatorname{rank} \mathcal{E}'' = h^{3,0}(\ell-1) \\ \operatorname{rank} \mathcal{E}''' = \delta \end{cases}$$

The first two are "understood". The third is not. Simplest phenomenon consistent with Hilbert scheme of W's with

$$h\zeta = [W - H]$$

having multiple components would have been a correction term $\delta \geqq 0$ with $\delta = 0$ for "good" components ($L \gg 0$). This not the case.

$$\rightsquigarrow \ \operatorname{sing}^\vee Z = \Lambda^\perp \subset \check{\mathbb{P}}^3$$



$\mathbb{R}^2 \cap D =$ 

(quasi-local normal crossings)

<u>Note</u>: Have vanishing cycles

$\delta_1, \delta_2, \delta_3$ with

$$\delta_1 + \delta_2 + \delta_3 \sim 0$$

HC $\Rightarrow$ this is general phenomenon