Fourier-Mukai Transforms for Singular Schemes and Geometric Applications

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$$\operatorname{Hom}^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \operatorname{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[i])\,.$$

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- ${\mathcal{F}^\bullet}^\vee = \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{O}_X)$ is the dual in the derived category.
- If $f\colon X \to Y$ is a proper morphism, $f^!\mathcal{O}_Y$ is the relative dualizing complex. When Y is a point, write $\mathcal{D}^\bullet_X$.

Birational Geometry

A birational map $f\colon X \dashrightarrow X^+$ between $\mathbb{Q}$-Gorenstein varieties is a *generalized flip* if there is a diagram

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\begin{array}{ccc}
 & Z & \\
{}^{\pi}\swarrow & & \searrow{}^{\pi^+} \\
X & & X^+
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of birational maps with Z smooth and the divisor $D = \pi^*(K_X) - \pi^{+*}(K_{X^+})$ is effective.
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When $D = 0$, $f\colon X \dashrightarrow X^+$ is a *generalized flop*.
Conjecture (Bondal, Orlov): If f is a generalized flip of smooth projective varieties, there is a fully faithful functor $F\colon D^b_c(X^+) \to D^b_c(X)$. For generalized flops, F is an equivalence of categories (K-equivalence $\implies$ D-equivalence).

Birational Geometry

Example (Bondal, Orlov): Suppose that X is a smooth algebraic variety containing a subvariety $Y \simeq \mathbb{P}^k$ with normal bundle $\mathcal{O}(-1)^{\oplus l+1}$ where $l \leq k$. Let $\tilde{X}$ be the blow-up of X along Y and X^+ its blow-down such that the exceptional divisor $\tilde{Y} \simeq \mathbb{P}^k \times \mathbb{P}^l$ projects onto $\mathbb{P}^l$. Then one has the flip

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{\pi}\swarrow & & \searrow^{\pi^+} \\ X & & X^+ \end{array}$$

It is a flop if $l = k$. The functor

$$\mathbf{R}\pi_*\mathbf{L}\pi^{+*} \colon D^b_c(X^+) \to D^b_c(X)$$

is fully faithful. It is an equivalence if $l = k$.

Birational Geometry

In some good cases, for instance,

- Smooth projective threefolds. *(Kollár)*
- Projective threefolds with $\mathbb{Q}$-factorial terminal singularities. *(Kawamata)*

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A *flop* is a diagram

$$\begin{array}{ccc} X & & X^+ \\ & \searrow^{\pi} \quad {}^{\pi^+}\swarrow & \\ & W & \end{array}$$

with π and π^+ isomorphisms in codimension one, K_X and K_{X^+} $\mathbb{Q}$-trivial on the fibers of π and π^+ respectively and there is a $\mathbb{Q}$-Cartier divisor D on X relatively ample for π and with $-D$ relatively ample for π^+.

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Conjecture: Let W be a projective variety with Gorenstein singularities . If

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are crepant resolutions, then $D^b_c(X) \simeq D^b_c(Y)$.

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- In dimension 3 and for smooth projective varieties. *(Bridgeland)*.
- In dimension 3 and for quasiprojective varieties with terminal Gorenstein singularities.*(Chen)*.
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Conjecture: If $G \subset SL(n, \mathbb{C})$ is finite and $Y \to \mathbb{C}^n/G$ is a crepant resolution, then $D^b_c(Y) \simeq D^b_G(\mathbb{C}^n)$. *(Reid)*

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- *Orlov's representation theorem:*
  If X and Y are smooth projective varieties, for any exact fully faithful functor $F\colon D^b_c(X)\to D^b_c(Y)$ there is $\mathcal{K}^\bullet\in D^b_c(X\times Y)$ such that $F\simeq\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ where

  $$\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{G}^\bullet)=\mathbf{R}\pi_{Y*}(\pi_X^*\mathcal{G}^\bullet\overset{\mathbf{L}}{\otimes}\mathcal{K}^\bullet)$$

  $\pi_X\colon X\times Y\to X$ and $\pi_Y\colon X\times Y\to Y$ being the projections.

The functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is called the integral functor of kernel $\mathcal{K}^\bullet$.

The smooth case

- *Bondal and Orlov's criterion of fully faithfulness:*
  If $\mathrm{ch}(k) = 0$, then the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is fully faithful if and only if the kernel $\mathcal{K}^\bullet$ is *strongly simple* over X, that is,
  1. $\mathrm{Hom}^i_{D(Y)}(\mathbf{L}j^*_{x_1}\mathcal{K}^\bullet, \mathbf{L}j^*_{x_2}\mathcal{K}^\bullet) = 0$ unless $x_1 = x_2$ and $0 \leq i \leq \dim X$;
  2. $\mathrm{Hom}^0_{D(Y)}(\mathbf{L}j^*_x\mathcal{K}^\bullet, \mathbf{L}j^*_x\mathcal{K}^\bullet) = k$ for every point x.

  $j_x \colon \{x\} \hookrightarrow X$ being the natural embedding.

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f Cohen-Macaulay $\Leftrightarrow f^!\mathcal{O}_S \simeq \mathcal{D}_f[m]$ with $m = \dim f$.

f Gorenstein $\Leftrightarrow f$ Cohen-Macaulay and $\mathcal{D}_f = \omega_f$ line bundle.

The smooth case

The proof of Bondal and Orlov's criterion is based on:

Key Proposition: Let $j\colon Y \hookrightarrow X$ be a closed immersion of codimension $d = m - n$ of smooth varieties and $0 \neq \mathcal{K}^\bullet \in D^b_c(X)$. If $\mathbf{L}_i j_x^* \mathcal{K}^\bullet = \operatorname{Hom}^{m-i}_{D(X)}(\mathcal{O}_x, \mathcal{K}^\bullet) = 0$ unless $x \in Y$ and $i \in [0, d]$, then $\mathcal{K}^\bullet \simeq \mathcal{K}$ is a sheaf on X and $\operatorname{supp} \mathcal{K} = Y$ (topologically).

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$$\operatorname{Hom}^i_{D(X)}(\mathcal{O}_{x_1}, \Phi_{X \to X}^{\mathcal{M}^\bullet}(\mathcal{O}_{x_2})) \simeq \operatorname{Hom}^i_{D(Y)}(\Phi(\mathcal{O}_{x_1}), \Phi(\mathcal{O}_{x_2}))$$

$$\Phi(\mathcal{O}_x) \simeq L j_x^* \mathcal{K}^\bullet$$

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Remark: The argument with the Kodaira-Spencer map uses essentially that k has characteristic zero.

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(joint with Hernández-Ruipérez, and Sancho de Salas, F.)
Let $X \to S$ and $Y \to S$ be proper morphisms and
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Remark: fhd (resp. fpd) weaker than finite Tor-amplitude (resp. Ext-amplitude) *(Kuznetsov)*.

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- If $f\colon Z \to T$ is projective and $\mathcal{O}(1)$ relatively very ample, the following conditions are equivalent:
  1. $\mathcal{E}^\bullet$ has fhd over T.
  2. $\mathcal{E}^\bullet$ has fpd over T.
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  3. $\mathbf{R}f_*(\mathcal{E}^\bullet(r))$ is a perfect complex for every $r \in \mathbb{Z}$.
- If $f\colon Z \to T$ is locally projective and $\mathcal{E}^\bullet$ has fhd over T

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, f^!\mathcal{O}_T) \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathcal{G}^\bullet \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, f^!\mathcal{G}^\bullet)$$

  for $\mathcal{G}^\bullet$ in $D^b_c(T)$.

Boundness conditions

- If $f\colon Z \to T$ is locally projective and $\mathcal{E}^\bullet$ has fhd over T, then $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, f^!\mathcal{O}_T)$ has fhd over T and

$$\mathcal{E}^\bullet \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, f^!\mathcal{O}_T), f^!\mathcal{O}_T)$$

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- Suppose that $X \to S$ is locally projective and $\mathcal{K}^\bullet \in D^b_c(X \times_S Y)$. Then,
  1. $\mathcal{K}^\bullet$ has fhd over $X \Leftrightarrow \Phi^{\mathcal{K}^\bullet}_{X\to Y}$ maps $D^b_c(X)$ to $D^b_c(Y)$.
  2. If $\mathcal{K}^\bullet$ has fhd over X and Y, the functor $\Phi^{\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times_S Y}}(\mathcal{K}^\bullet, \pi_Y^!\mathcal{O}_Y)}_{Y\to X}\colon D^b_c(Y) \to D^b_c(X)$ is a right adjoint to $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X) \to D^b_c(Y)$.

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Key Proposition: Let $j\colon Y \hookrightarrow X$ be a closed immersion of codimension $d = m - n$ of pure dimensional CM schemes and $\mathcal{K}^\bullet \in D^b_c(X)$. If for all $x \in X$ there is Z_x with

$$\operatorname{Hom}^i_{D(X)}(\mathcal{O}_{Z_x}, \mathcal{K}^\bullet) = 0$$

unless $x \in Y$ and $i \in [n, m]$, then $\mathcal{K}^\bullet \simeq \mathcal{K}$ is a sheaf on X whose topological support is contained in Y.

Criterion in $\mathrm{ch}(k) = 0$

Let X and Y be proper schemes. When X is CM, an object $\mathcal{K}^\bullet \in D^b_c(X \times Y)$ is strongly simple over X if :

1. For every $x \in X$ there is Z_x such that $\mathrm{Hom}^i_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_{x_1}}), \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{x_2})) = 0$ unless $x_1 = x_2$ and $0 \leq i \leq \dim X$;
2. $\mathrm{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x), \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)) = k$ for every x.

Criterion in $\mathrm{ch}(k) = 0$

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Criterion: If $\mathrm{ch}(k) = 0$, X is projective CM and integral and $\mathcal{K}^\bullet \in D^b_c(X \times Y)$ has fhd over X and Y, then

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Remark: In the smooth case, this improves the characterization of Bondal and Orlov.

Counterexample

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Since $F_*(\mathcal{O}_x) = \mathcal{O}_{F(x)} \implies \Gamma$ is strongly simple over X.
But $F_*(\mathcal{O}_X)$ is locally free of rank $p^m \implies$
$\mathrm{Hom}^0(F_*(\mathcal{O}_X), \mathcal{O}_{F(x)}) \simeq k^{p^m}$ whereas $\mathrm{Hom}^0(\mathcal{O}_X, \mathcal{O}_x) \simeq k$.
Then F_* is not fully faithful.

In arbitrary characteristic

There is a similar criterion: the conditions on $\mathcal{K}^\bullet$ for $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ to be fully faithful are *the CM orthonormality conditions* over X:

1. The same orthogonality condition than if $\mathrm{ch}(k) = 0$.
2. There is a point $x \in X$ such that one of the following properties is fulfilled:
   a) $\mathrm{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_X), \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)) \simeq k$.
   b) $\mathrm{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_x}), \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)) \simeq k$ for any Z_x.
   c) $\dim_k \mathrm{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_x}), \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_x})) \leq l(\mathcal{O}_{Z_x})$ for any Z_x, where $l(\mathcal{O}_{Z_x})$ is the length of $\mathcal{O}_{Z_x}$.

Moreover, the requirement "X projective, CM and integral" can be relaxed to "X projective, CM, connected and equidimensional".

Relative setting

Let $X \to S$ and $Y \to S$ be proper and *flat* morphisms, $\mathcal{K}^\bullet \in D^b_c(X \times_S Y)$ and $\Phi = \Phi^{\mathcal{K}^\bullet}_{X \to Y}$. Denote X_s, Y_s the fibers over $s \in S$ and $\Phi_s \colon D^-(X_s) \to D^-(Y_s)$ the induced absolute functor of kernel $\mathcal{K}^\bullet{}_s$.

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Suppose that $X \to S$ is locally projective and $\mathcal{K}^\bullet$ has fhd over X and Y.

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- Φ is fully faithful (resp. equivalence) $\Leftrightarrow$ Φ_s is fully faithful (resp. equivalence) for all $s \in S$.
- We obtain straightforward extensions of the absolute criteria for
  - $\mathrm{ch}(k) = 0$ and $X \to S$ with integral CM fibers.
  - arbitrary characteristic and $X \to S$ with connected equidimensional CM fibers.

Criteria for equivalence

Suppose that X and Y are projective and Gorenstein. The funtor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ has two nice adjoints

$$H = \Phi^{\mathcal{K}^{\bullet\vee}\otimes\pi^!_Y\mathcal{O}_Y}_{Y\to X} \text{ and } G = \Phi^{\mathcal{K}^{\bullet\vee}\otimes\pi^!_X\mathcal{O}_X}_{Y\to X}$$

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$\Phi^{\mathcal{K}^\bullet}_{X\to Y}$equivalence $\Leftrightarrow$ $\mathcal{K}^\bullet$satisfies CM conditions over X and Y .

Since $\{\mathcal{O}_x\}$ and $\{\mathcal{O}_{Z_x}\}$ are spanning classes for $D^b_c(X)$, generalizing the *Bridgeland's result*, one has:

A fully faithful functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is equivalence if and only if one of the following conditions is fulfilled:

1. $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x) \simeq \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x) \otimes \omega_Y$ for all x.
2. $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_x}) \simeq \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_x}) \otimes \omega_Y$ for all x and all Z_x.

Integral elliptic fibrations

Let $X \to S$ be an integral elliptic fibration (proper Gorenstein morphism whose fibers are integral curves of genus one).

Let $\hat{X} \to S$ be its dual fibration (relative moduli space of torsion free rank 1 sheaves of degree 0).

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The Poincaré sheaf is

$$\mathcal{P} = \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}_X(H) \otimes \pi_2^* \mathcal{O}_X(H) \otimes \rho^* \omega^{-1} ,$$

where π_i are the projections, $\rho\colon X \times_S X \to S$, $H = \sigma(S)$ and $\omega = R^1 p_* \mathcal{O}_X$.
Easy application: $\Phi^{\mathcal{P}}_{X \to X} \colon D^b_c(X) \to D^b_c(X)$ is an equivalence of categories.

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Let $\mathcal{I}_\Delta \in D^b_c(X \times_S X)$ be the ideal sheaf of the diagonal immersion $\delta \colon X \hookrightarrow X \times_S X$.
$\mathcal{I}_\Delta$ has fhd over both factors and $\mathcal{I}_{\Delta_s}$ satisfies the CM orthonormality conditions over both factors for every $s \in S$.
Theorem: The functor $\Phi^{\mathcal{I}_\Delta}_{X \to X} \colon D^b_c(X) \to D^b_c(X)$ is an equivalence of categories.

FM partners

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FM partners share many geometrical properties.
What happens in the singular case?

Geometric consequences

Let X and Y be projective Gorenstein schemes. If the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X) \xrightarrow{\sim} D^b_c(Y)$ is an equivalence, then:

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- For every integer i, $H^0(X, \omega_X^i) \simeq H^0(Y, \omega_Y^i)$, so that $\kappa(X) = \kappa(Y)$.

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Next Question: Are the CM and Gorenstein conditions invariant under Fourier-Mukai transforms?

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- The right adjoint is integral and its kernel $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times_S Y}}(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y)$ has fhd over X and Y.
- There is an isomorphism

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{K}^\bullet, \pi_X^! \mathcal{O}_X) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y)\,.$$

(in the smooth case, this is the commutation of an equivalence with Serre functors)

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For projective schemes the Cohen-Macaulay and the Gorenstein condition can be recovered from their derived category.

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Theorem:

Let X be a projective Cohen-Macaulay scheme of dimension m and Y a FM partner of X. Assume that Y is projective.

1. If Y is reduced, then Y is equidimensional of dimension m.
2. If Y is equidimensional and $\dim Y = m$, then Y is Cohen-Macaulay. Moreover, if X is Gorenstein, then Y is Gorenstein as well.

Geometric consequences

Idea of the proof:

Let $\Phi = \Phi^{\mathcal{K}^\bullet}_{X\to Y} : D^b_c(X) \xrightarrow{\sim} D^b_c(Y)$ be the FM functor.

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Let $\Phi = \Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X) \xrightarrow{\sim} D^b_c(Y)$ be the FM functor.
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$$\Psi_1 = \Phi^{\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{K}^\bullet, \pi^!_X \mathcal{O}_X)}_{X\to Y}\colon D^b_c(X) \to D^b_c(Y)$$

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are isomorphic.
For any Z_x and any point $y \in Y$, one has

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y, \Psi_1(\mathcal{O}_{Z_x})) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y, \Psi_2(\mathcal{O}_{Z_x}))\,.$$

Geometric consequences

If $\Phi_{Z_x}\colon D^b_c(Z_x)\to D^b_c(Y)$ is the functor of kernel $\mathbf{L}j^*_{Z_x}\mathcal{K}^\bullet$, computing we get:

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\Phi_{Z_x}(\omega_{Z_x}),\mathcal{O}_y)\simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\Phi_{Z_x}(\mathcal{O}_{Z_x}),\mathcal{O}_y^\vee)[m]\,,$$

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Since X is CM, $\{\Phi(\mathcal{O}_{Z_x})\}$ is a spanning class for $D^b_c(Y)$. Taking Z_x such that $y\in\operatorname{supp}(\Phi(\mathcal{O}_{Z_x}))$, the complex $\mathcal{Q}^\bullet=\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\Phi_{Z_x}(\mathcal{O}_{Z_x}),\mathcal{O}_y^\vee)[m]$ satisfies

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(1) If Y is reduced, take y a smooth point of a component Y_0. Since $\mathcal{O}_y^\vee\simeq\mathcal{O}_y[\dim Y_0]$, this gives $\dim Y=m$.
(2) If j_0 is the depth of $\mathcal{O}_{Y,y}$, we get $j_0\geq m$. Then Y CM.

Geometric consequences

If X is Gorenstein, then $\omega_{Z_x} \simeq \mathcal{O}_{Z_x}$ and we deduce

$$\operatorname{Hom}_Y(\mathcal{H}^0(\Phi_{Z_x}(\mathcal{O}_{Z_x})),\mathcal{O}_y) \simeq \operatorname{Hom}_Y(\mathcal{H}^0(\Phi_{Z_x}(\mathcal{O}_{Z_x})), \mathcal{E}xt^m_{\mathcal{O}_Y}(\mathcal{O}_y, \mathcal{O}_Y))\,.$$

Thus, $\dim \mathcal{E}xt^m_{\mathcal{O}_Y}(\mathcal{O}_y, \mathcal{O}_Y) = 1$ and Y is Gorenstein.